

If ω is a k -form, redefine the $(k+1)$ -form $d\omega$ by

$$(d\omega)(u, v_0, \dots, v_k) = \sum_{\ell=0}^k (-1)^\ell (D\omega)(u, v_\ell, v_0, v_1, \dots, \hat{v}_\ell, \dots, v_k).$$

(Here the \wedge over v_ℓ means that v_ℓ is omitted.) We claim this $d\omega$ is the same as the $d\omega$ defined previously. This is checked by showing that this $d\omega$ is linear and alternating in the v_0, \dots, v_k , and has the same values on the basis elements of $V \times V \times \dots \times V$ as the old $d\omega$. The linearity is clear, given our comments regarding the operator D ; $d\omega$ is alternating since computation shows that it vanishes when any two successive arguments are equal. Suppose now ω is a one-form; $\omega = \sum w_i dq^i$, where $\{q^i\}$ are coordinates on M and $\{e_i\}$ are the corresponding basis elements of $V \cong T_u(M)$. Then $w_i(u) = \omega(u, e_i)$. By our old definition

$$d\omega = \sum_{i < j} \left(\frac{\partial w_j}{\partial q^i} - \frac{\partial w_i}{\partial q^j} \right) dq^i \wedge dq^j = \sum_{i < j} d\omega(u, e_i, e_j) dq^i \wedge dq^j.$$

To prove that the two definitions coincide for one-forms it will thus suffice to show that $d\omega(u, e_i, e_j)$ is the same as in the new definition.

But in the new definition

$$d\omega(u, e_i, e_j) = D\omega(u, e_i, e_j) - D\omega(u, e_j, e_i),$$

and

$$Df(u, e_i) = \partial f / \partial q_i.$$

Hence

$$d\omega(u, e_i, e_j) = \frac{\partial \omega(u, e_j)}{\partial q^i} - \frac{\partial \omega(u, e_i)}{\partial q^j} = \frac{\partial w_j}{\partial q^i} - \frac{\partial w_i}{\partial q^j},$$

which is what we were trying to prove. Similar techniques show that the two definitions are the same for general k -forms.

We are now ready to define the map s which makes a $(p-1)$ -form out of every p -form. If ω is a k -form, let

$$(s\omega)(u; v_1, \dots, v_{k-1}) = \int_0^1 t^{k-1} \omega(tu; u, v_1, \dots, v_{k-1}) dt.$$

Here we consider the open set U as part of the vector space $V = \mathbb{R}^n$, which has also been identified with $T_p(U)$. Thus on the right-hand side of the equation, the second argument, $u \in U$, is viewed as a vector of V . But since U is an open ball, tu , the first argument, is in U for all $t \neq 1$. It is now easy to check that $s\omega$ is a $(k-1)$ -form -- linear, alternating, and smooth as a function of u .

We now take a k -form ω and show, at last, that $ds(\omega) + sd(\omega) = \omega$.

First,

$$\begin{aligned} D(s\omega)(u, v, v_1, \dots, v_{k-1}) &= \int_0^1 D[t^{k-1} \omega(tu, v, u, v_1, \dots, v_{k-1})] dt \\ \text{(since all functions involved are smooth and bounded)} &= \int_0^1 t^k D\omega(tu, v, u, v_1, \dots, v_{k-1}) dt \\ &\quad + \int_0^1 t^{k-1} \omega(tu, v, v_1, \dots, v_{k-1}) dt. \end{aligned}$$

The latter term appears as it does since ω is linear in the third variable, and it was proved that if f is linear, $Df(u, v) = f(v)$. Now

$$\begin{aligned} d(s\omega)(u, v_1, \dots, v_k) &= \sum_{\ell=1}^k (-1)^{\ell-1} D(s\omega)(u, v_\ell, v_1, \dots, \hat{v}_\ell, \dots, v_k) \\ &= \sum_{\ell=1}^k (-1)^{\ell-1} \left[\int_0^1 t^k D\omega(tu, v_\ell, u, v_1, \dots, \hat{v}_\ell, \dots, v_k) dt \right. \\ &\quad \left. + \int_0^1 t^{k-1} \omega(tu, v_\ell, v_1, \dots, \hat{v}_\ell, \dots, v_k) dt \right], \end{aligned}$$

and

$$\begin{aligned} s(d\omega)(u, v_1, \dots, v_k) &= \int_0^1 t^k d\omega(tu, u, v_1, \dots, v_k) dt \\ &= \int_0^1 \sum_{\ell=1}^k (-1)^\ell t^k D\omega(tu, v_\ell, u, v_1, \dots, \hat{v}_\ell, \dots, v_k) dt \\ &\quad + \int_0^1 t^k D\omega(tu, u, v_1, \dots, v_k) dt. \end{aligned}$$

When we add $d(s\omega)$ and $s(d\omega)$, the first terms of each expression cancel;

also,

$$\begin{aligned} &\sum_{\ell=1}^k (-1)^{\ell-1} \int_0^1 t^{k-1} \omega(tu, v_\ell, v_1, \dots, \hat{v}_\ell, \dots, v_k) dt \\ &= \sum_{k=1}^{\ell} (-1)^{\ell-1} \int_0^1 (-1)^{\ell-1} t^{k-1} \omega(tu, v_1, \dots, v_k) dt \quad \text{since } \omega \text{ is alternating} \\ &= k \int_0^1 t^{k-1} \omega(tu, v_1, \dots, v_k) dt. \end{aligned}$$

Hence

$$\begin{aligned} (sd\omega + ds\omega)(u, v_1, \dots, v_k) &= \int_0^1 [t^k D\omega(tu, u, v_1, \dots, v_k) + kt^{k-1} \omega(tu, v_1, \dots, v_k)] dt \\ &= \int_0^1 \frac{d}{dt} [t^k \omega(tu, v_1, \dots, v_k)] dt \end{aligned}$$

