

"smooth"; to be precise about this, we regard X as a map of U into $T(U)$. Since $Xa \in T_a(U)$, we can write X as $a \rightsquigarrow (a, Xa)$.

Definition. A vector field is a smooth function $X: U \rightarrow T(U)$ such that the composite $\pi \circ X$ is the identity function: $\pi \circ X = 1_U$.

The last requirement means simply that

$$X: u \rightsquigarrow (u, \text{some tangent vector at } u).$$

Recall that the vector space $T^a U$ has the basis $d_a q^1, \dots, d_a q^n$, where the q^i are coordinates in U . It seems natural to put coordinates on $T(U)$ so that the point $(a, w) \in T(U)$ is viewed as $(q^1, \dots, q^n, h^1 d_a q^1, \dots, h^n d_a q^n)$, where the q^i and the h^i are the coordinates of a and w , respectively. Thus $w = h^1 d_a q^1 + \dots + h^n d_a q^n$. In a similar way, we put $2n$ coordinates on the tangent bundle. We have seen that, if f is a function on U , then

$$d_a f = \sum \left(\frac{\partial f}{\partial q^i} \right)_a d_a q^i$$

is a point of $T^a U$. Hence we can write

$$(*) \quad df = \sum \left(\frac{\partial f}{\partial q^i} \right) dq^i$$

for the function which assigns to every point a of U the cotangent vector $d_a f$; it is thus the cotangent-space analogue of a vector field. Expressions like the above constantly pop up in physics; for example, if the three components of force in space are represented by the functions F_i of x, y , and z , the infinitesimal work is usually defined by

$$dW = F_1 dx + F_2 dy + F_3 dz.$$

When we interpret dW, dx , and so forth, not as infinitesimals, but as cotangent vector fields, this equation resembles equation (*). This will actually be the way we put the notion of "infinitesimal" on a sound mathematical basis.

§4 Vector Bundles

Take a long narrow strip of paper, draw a line down the middle lengthwise, and paste the ends together by twisting once to form a figure called the möbius strip. The line down the middle now becomes a circle, and the surface can be thought of as composed of vectors ("fibers") perpendicular to the circle and radiating from it. We are about to see how the mobius strip can be viewed as part of a vector bundle over the circle.

First we define the simpler notion of pre-bundle. Given two sets U and V , their cartesian product $U \times V$ is defined to be the set of all pairs (u, v) with $u \in U, v \in V$. Given sets G, E, U , and maps σ and π forming the following diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow \pi \\ G & \xrightarrow{\sigma} & U \end{array}$$

we define the pullback $G \times_U E$ as the universal object making the square

$$\begin{array}{ccc} G \times_U E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ G & \longrightarrow & U \end{array}$$

commutative; (universal objects and commutative squares will be defined in

due course). Specifically, the pullback can be defined as the following set:

$$G \times_U E = \{(e, g) \mid e \in E, g \in G, \pi e = \sigma g\}.$$

A pre-bundle, usually written $\begin{array}{c} B \\ \downarrow \pi \\ U \end{array}$, is a pair of sets B, U and a map

between them such that B and U are open in some Euclidean space, π is a smooth map, and on each fiber $\pi^{-1}(u) = \{b \mid b \in B, \pi b = u\}$ there is a smooth addition and a scalar multiplication defined so that each fiber forms a vector space. By this we mean that

(a) if $b_1, b_2 \in B$ with $\pi(b_1) = \pi(b_2)$ (i. e., b_1 and b_2 are in the same fiber), and if k is a real number, then there are points $b_1 + b_2$ and $b_1 k$ in B with $\pi(b_1 + b_2) = \pi(b_1 k) = \pi(b_1)$,

(b) the resulting operations on each fiber satisfy the vector space axioms, so that each fiber by itself becomes a real vector space,

(c) the maps $+: B \times_U B \rightarrow B$ and $\cdot : B \times \mathbb{R} \rightarrow B$ are smooth.

To state property (c) we must make the additional postulate that $B \times_U B$, the

pullback in the diagram $\begin{array}{ccc} B \times_U B & \longrightarrow & B \\ \downarrow & & \downarrow \pi \\ B & \xrightarrow{\pi} & U \end{array}$ may be embedded in some

Euclidean space as an open set. U is usually called the base space, B the total space, and π the projection of B on U . Notice that while U is an open subset of some vector space \mathbb{R}^k , U is not itself a vector space.

Neither is B ; instead, it is made up of a lot of vector spaces glued together like a bundle of sticks.

The easiest example of a pre-bundle is the product space $U \times V$, where U is an open set in \mathbb{R}^n and V is a real vector space, $V = \mathbb{R}^m$ for some m . When we define $\pi(u, v) = u$, we see that the fiber $\pi^{-1}(a) = \{(a, v) \mid v \in V\}$ is just a copy of V , and so inherits the vector-space structure of V . One can check that the other requirements are satisfied. Such a pre-bundle is called a local vector bundle.

Suprisingly enough, the tangent bundle to an open set U is a pre-bundle, (indeed, a local vector bundle) with U the base space, $T(U)$ the total space, and $\pi: (a, v) \rightsquigarrow v$ the projection. Moreover, there is an isomorphism $T(U) \xrightarrow{\cong} U \times \mathbb{R}^n$ obtained by mapping $(a, v) \rightsquigarrow (a, (v^1, \dots, v^n))$, where v has coordinates v^i ; that is,

$$v = v^1 \left(\frac{\partial}{\partial q^1} \right)_a + \dots + v^n \left(\frac{\partial}{\partial q^n} \right)_a .$$

This isomorphism is a bundle map, in the sense that points lying in the fiber over a point a of U are mapped into points in the same fiber. More generally, the pair of smooth maps (H, h) in the diagram

$$\begin{array}{ccc} B & \xrightarrow{H} & \tilde{B} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ U & \xrightarrow{h} & \tilde{U} \end{array}$$

is called the bundle map if $\tilde{\pi}H = h\pi$ (that is, if the diagram commutes), and if the map H is linear on each fiber. The first condition means that the fiber over a is mapped into the fiber over $h(a)$; the second can be stated

$$\pi b_1 = \pi b_2 \implies H(b_1 + b_2) = H(b_1) + H(b_2), \quad H(bk) = H(b)k.$$

A cross-section of a pre-bundle is a smooth map $\chi: U \rightarrow B$ such that $\pi\chi = 1_U$; in other words, $\chi(u)$ lies in the fiber over u . For example, we

always have the zero section: $\chi(u) = 0$, all $u \in B$. In the case of a local vector bundle, for any constant vector v_0 of V the map $u \mapsto (u, v_0)$ is a cross-section. In general, the set of all cross-sections of a bundle π , denoted by $\Gamma(\pi)$, has an additive structure $[(\chi_1 + \chi_2)(u) = \chi_1(u) + \chi_2(u)]$ as well as right multiplication not just by ordinary scalars but by real-valued smooth functions: if $f \in \mathcal{F}$, $(\chi f)(u) = \chi(u) \cdot f(u)$. Since χf and $\chi_1 + \chi_2$ are also smooth, we find that $\Gamma(\pi)$ forms a module over \mathcal{F} .

Now that we know that $T.(U)$ is a pre-bundle, we can redefine a vector field to be a cross-section of $T.(U)$. The inevitable dual object, a cross-section of $T^*(U)$, will be called a differential one-form, or a co-vector field. Before considering these creatures, we observe that an ordinary vector field may act on functions much like a directional derivative; that is, given a function and a vector field, we define a new function whose value at every point is the derivative of the original function in the direction given by the vector field at that point. Formally, given $g \in \mathcal{F}$ and $X \in \Gamma(\pi)$, define

$$(Xg)(a) = \langle d_a g, Xa \rangle .$$

It is important not to confuse Xg , which is a function, with the vector field Xg , the product of the cross-section X of $T.(U)$ with the function g . The context should usually make clear which one is meant.

Since $T^*(U)$ is a pre-bundle in exactly the same way that $T.(U)$ was, it makes sense to speak of a cross-section of $T^*(U)$, and such a function ω will be called a differential one-form on U . For example, if f is a smooth function taking U into \mathbb{R} , the function df defined by $df(a) = (a, d_a f) \in T^*(U)$

is a differential one-form. From the i^{th} -coordinate function q^i , which maps a point of U to its i^{th} -coordinate, we get a form dq^i , and we can express df for a general f in terms of this basis:

$$(1) \quad df(a) = (q^1(a), \dots, q^n(a), \left(\frac{\partial f}{\partial q^1}\right)_a dq^1, \dots, \left(\frac{\partial f}{\partial q^n}\right)_a dq^n).$$

The differential one-forms are a module over \mathcal{F} , just as the vector fields were; given a form ω and a function $g \in \mathcal{F}$, we define $g\omega$ (a new form) by the equation $g\omega(a) = (a, g(a)\omega_a)$ if $\omega(a) = (a, \omega_a)$. Then formula (1) shows that

$$df = \sum \left(\frac{\partial f}{\partial q^i}\right) dq^i,$$

which is now a rigorously based statement of the familiar law of partial differentiation.

It is easy to see that any differential one-form on U can be written as $\omega = \sum_{i=1}^n h^i dq^i$, where the h^i are n smooth functions on U . But it is an important fact that in general not every form ω is of the form df for some smooth f . This problem is related to the well-known condition for exactness of a first-order differential equation (cf. Apostol, vol. II, p. 239). Thus we can define a one-form ω to be exact if there is a smooth function f on U such that $\omega = df$. In a simply-connected two-dimensional region, like the interior of a circle, a necessary and sufficient condition for the one-form $\omega = h dx + k dy$ to be exact is that $\partial h / \partial y = \partial k / \partial x$. In general, however, this is only a necessary condition (cf. Mackey, p. 18, footnote), so we define a new expression, the differential of ω , as

$$d\omega = \sum_{\substack{i,j=1 \\ i < j}}^n \left(\frac{\partial h^i}{\partial q^j} - \frac{\partial h^j}{\partial q^i} \right) dq^i dq^j \quad \text{if } \omega = \sum h^i dq^i$$

and we say that ω is closed if $d\omega = 0$. Thus any exact form is closed but not vice versa. The proverbial "alert reader" may have noticed that the above conclusion depends on the definition of exactly what $dq^i dq^j$ means; for the moment, however, we merely point out the analogy between the form of $d\omega$ and that of the "curl" of a vector field in three-space, which allows us to compare and contrast our condition for a form to be exact with the commonplace of physics that a field has a potential function if and only if its curl is zero. (It is worth pointing out that in n -space there are $\frac{n(n-1)}{2}$ different $dq^i dq^j$ for i and j ranging between 1 and n , $i < j$; when $n = 3$ it happens that $\frac{1}{2}n(n-1)$ is also 3. So while in the three-dimensional case the curl of a field is a vector of the same dimension as the field, this need not be true in general.)

We previously considered the effect of a mapping $\varphi: U \rightarrow \tilde{U}$ on cotangent vectors; for each point a of U , φ induces a linear map $\varphi^*: T^{\varphi(a)} \tilde{U} \rightarrow T^a U$. It is easy to generalize this to the case of differential forms. Let $\tilde{\omega}$ be a one-form on \tilde{U} ; we desire to construct a form $\varphi^* \tilde{\omega}$ on U . Given $a \in U$, we have $\varphi(a) \in \tilde{U}$. If $\tilde{\omega}(\varphi(a)) = (\varphi(a), \omega)$, $\omega \in T^{\varphi(a)}(\tilde{U})$, then define $(\varphi^* \tilde{\omega})(a) = (a, \varphi^*(\omega))$ where $\varphi^*(\omega) \in T^a(U)$, so $(a, \varphi^*(\omega)) \in T \cdot (U)$. In a typical physical application, $\tilde{\omega}$ would represent the work for some displacement, and $\varphi^* \tilde{\omega}$ would be the same work function expressed in terms of a different generalized coordinate system, given by the change φ of coordinates from U to \tilde{U} .

In the case where $\omega = d\tilde{f}$, where \tilde{f} is a real-valued smooth function on \tilde{U} , we can derive

$$\varphi^*(d\tilde{f}) = d(\tilde{f} \circ \varphi) = \sum_{j=1}^n \frac{\partial(\tilde{f} \circ \varphi)}{\partial q^j} dq^j.$$

In particular,

$$\varphi^*(d\tilde{q}^i) = \sum_{j=1}^m \frac{\partial(\tilde{q}^i \circ \varphi)}{\partial q^j} dq^j.$$

Here the coefficients of dq^j form the familiar Jacobian matrix $\frac{\partial(\tilde{q}^i \circ \varphi)}{\partial q^j}$.

Moreover, $\varphi^*(\tilde{f}\omega) = (\varphi^*\tilde{f})(\varphi^*\omega) = (\tilde{f} \circ \varphi)(\varphi^*\omega)$, where $\varphi^*\tilde{f} = \tilde{f} \circ \varphi$; and when we have defined the operator d on forms it will turn out that

$d(\varphi^*\omega) = \varphi^*(d\omega)$. These equations can be interpreted as properties of the

map $\varphi^*: \Gamma(T^*\tilde{U}) \rightarrow \Gamma(T^*U)$, where $\Gamma(T^*(U))$ is the \mathcal{F} -module of all

differential one-forms on U . Finally, if we have the diagram of maps

$$U_1 \xrightarrow{\varphi} U_2 \xrightarrow{\psi} U_3 \quad \text{then} \quad \Gamma(T^*U_3) \xrightarrow{\psi^*} \Gamma(T^*U_2) \xrightarrow{\varphi^*} \Gamma(T^*U_1) \quad \text{and}$$

we have $(\psi\varphi)^*\omega = \varphi^*(\psi^*(\omega))$. We say that $(\)^*$ is a "contravariant functor".

We can pull-back not only differential forms but also whole pre-bundles along the map φ . Specifically, given the situation

$$\begin{array}{ccc} & \tilde{B} & \\ & \downarrow \pi & \\ U & \xrightarrow{\varphi} & \tilde{U} \end{array} \quad \text{where } \tilde{B}$$

is a pre-bundle, construct a new set

$$B = \{(a, \tilde{b}) \mid a \in U, \tilde{b} \in \tilde{B}, \text{ and } \varphi a = \pi \tilde{b}\}.$$

(This set B is exactly the "pullback" $U \times_{\tilde{U}} \tilde{B}$ discussed earlier.) If we define $\pi(a, \tilde{b}) = a \in U$, then it can be shown that B is a pre-bundle. Intuitively speaking, given any point a of U we have found the fiber over the image of a under φ and made that fiber the fiber over a in our new bundle. Notice

that if \tilde{B} is a tangent bundle then its pullback need not be; for example, if \tilde{U} is some open set in the plane, and U is a line lying in \tilde{U} , the pullback B will have a two-dimensional vector space over each point of U , whereas the tangent bundle to a line is always one-dimensional.

§5. The Lagrange Equations

(As a reference for the following discussion of the Lagrange equations, cf. Goldstein, pp. 10-18, and Whittaker, pp. 30-35.)

We consider a system of N particles with coordinates x^1, \dots, x^{3N} , moving subject to constraints, in such a way that the system can be described by n generalized coordinates q^1, \dots, q^n . For example, the position of a sphere rolling on a plane is completely determined by x and y , the coordinates of the center of the sphere, and the three Eulerian angles θ, ψ, ϕ which determine how much the sphere rotates. We make the assumption that the constraints are holonomic; that is, we can make small displacements in each of the q^j independently. Given applied forces $F_i^{(a)}$ and constraint forces $F_i^{(c)}$ in each of the euclidean directions x^i , $i = 1, \dots, 3N$, and under the assumption that the forces of constraint do no work, we will show that the system obeys the Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial \mathcal{J}}{\partial \dot{q}^j} \right) - \frac{\partial \mathcal{J}}{\partial q^j} = Q_j, \quad j = 1, \dots, n$$

where $\mathcal{J} = \mathcal{J}(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$ is the kinetic-energy function, and Q_j is called the "generalized force in direction j " as defined below.

Newton's laws tell us exactly how the system behaves when viewed in the framework of euclidean space, \mathbb{R}^{3N} ; our problem is to describe it in the configuration space U whose coordinates are the q^j . Since we have an evident map $\varphi: U \rightarrow \mathbb{R}^{3N}$ which merely changes the q^j -description of a state of the system to the x^i -coordinates of \mathbb{R}^{3N} , a natural step is to try to "pull back" the laws of motion in \mathbb{R}^{3N} along φ . For example, if

$$W = \sum_i^{3N} (F_i^{(a)} + F_i^{(c)}) dx^i$$

is the differential form corresponding to work in euclidean space, then φ^*W gives the work in terms of the q^j . This is so because the change of the state of the system in time is described by a path in \mathbb{R}^{3N} ; but since we consider only motion subject to the constraints, each such path is the image under φ of a path in U . Now the "self-adjointness" rule given above for pullbacks makes it clear that $W^U = \varphi^*W$ has the same effect on a path of U as does W on the corresponding path in \mathbb{R}^{3N} . Computing,

$$\begin{aligned} W^U &= \sum_{i=1}^{3N} \varphi^*(F_i^{(a)} + F_i^{(c)}) \sum_{j=1}^n \frac{\partial x^i}{\partial q^j} dq^j \\ &= \sum_{j=1}^n \left(\sum_i F_i^{(a)} \frac{\partial x^i}{\partial q^j} \right) dq^j + \sum_{j=1}^n \left(\sum_i F_i^{(c)} \frac{\partial x^i}{\partial q^j} \right) dq^j, \end{aligned}$$

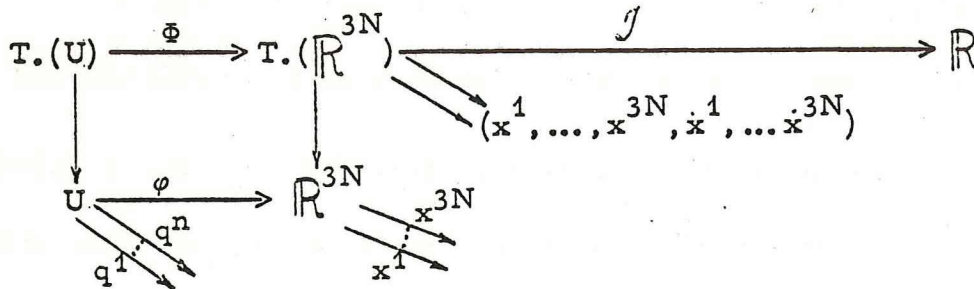
where we have identified F and $\varphi^*F = F \circ \varphi$ (that is, the force F as a function of the x^i and F viewed as a function of the q^i) as is the custom in physics.

Since we have assumed that the $F_i^{(c)}$ do no work, we get

$$W^U = \sum_{j=1}^n Q_j dq^j \quad \text{where} \quad Q_j = \sum_{i=1}^{3N} F_i^{(a)} \frac{\partial x^i}{\partial q^j}$$

are the generalized forces; "generalized" since they need not have the dimensions of force, although the product $Q_j q^j$ always has the dimensions of work.

The situation is now as follows



where the map Φ is defined by $\Phi(a, v) = (\varphi a, \varphi^* v)$. As above, we will follow the convention of writing $\frac{\partial J}{\partial q^i}$ when we really mean $\frac{\partial (J \circ \Phi)}{\partial q^i}$.

Then we have the following straightforward derivation of the Lagrange equations, starting from Newton's second law:

$$m_i \frac{d^2 x^i}{dt^2} = F_i^{(a)} + F_i^{(c)}, \quad i = 1, \dots, 3N;$$

therefore

$$(1) \quad \sum_{j=1}^n \left(\sum_{i=1}^{3N} m_i \frac{d^2 x^i}{dt^2} \frac{\partial x^i}{\partial q^j} \right) dq^j = \sum_{i=1}^{3N} (F_i^{(a)} + F_i^{(c)}) \sum_{j=1}^n \frac{\partial x^i}{\partial q^j} dq^j = \sum_{j=1}^n Q_j dq^j.$$

Now $\dot{x} = \frac{dx}{dt} = \sum_{j=1}^n \frac{\partial x}{\partial q^j} \dot{q}^j$. Therefore $\frac{\partial \dot{x}}{\partial \dot{q}^j} = \frac{\partial x}{\partial q^j}$. Hence

$$\begin{aligned} \frac{d^2 x}{dt^2} \frac{\partial x}{\partial q^j} &= \frac{d}{dt} (\dot{x}) \frac{\partial x}{\partial q^j} = \frac{d}{dt} \left(\dot{x} \frac{\partial x}{\partial q^j} \right) - \dot{x} \frac{d}{dt} \left(\frac{\partial x}{\partial q^j} \right) = \frac{d}{dt} \left(\dot{x} \frac{\partial \dot{x}}{\partial \dot{q}^j} \right) - \dot{x} \frac{\partial}{\partial q^j} (\dot{x}) \\ &= \frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}^j} \frac{\dot{x}^2}{2} \right) - \frac{\partial}{\partial q^j} \left(\frac{\dot{x}^2}{2} \right). \end{aligned}$$

$$\begin{aligned} \text{By (1), } \sum_{j=1}^n Q_j dq^j &= \sum_{j=1}^n \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}^i} \sum_i \frac{m_i \dot{x}_i^2}{2} - \frac{\partial}{\partial q^j} \sum_i \frac{m_i \dot{x}_i^2}{2} \right] dq^j \\ &= \sum_{j=1}^n \left(\frac{d}{dt} \left(\frac{\partial \mathcal{J}}{\partial \dot{q}^j} \right) - \frac{\partial \mathcal{J}}{\partial q^j} \right) dq^j . \end{aligned}$$

Since the dq^j are independent, we can equate the individual coefficients and get the desired Lagrange equations.

What is really going on here? A first clue is that our formula

$$\frac{dx}{dt} = \sum \frac{\partial x}{\partial q^j} \dot{q}^j$$

is just a special case of the formula in coordinates for

$T_a(\varphi)$, the map induced by φ on the tangent bundle. We have really "lifted"

the path $c: I \rightarrow U$, which describes the change of the system in time, to a

path $\tilde{c}: I \rightarrow T.(U)$, where $\tilde{c}(t) = (c(t), \dot{c}(t))$, and examined its form under

the map Φ . More pertinent is the reason why we have organized our equations

of motion in this form to begin with. Examining the derivation above shows

that we expressed Newton's second law, $m_i \frac{d^2 x^i}{dt^2} = F^i$, in terms of

$$\mathcal{J} = \sum \frac{1}{2} m_i (\dot{x}^i)^2, \text{ getting } \frac{d}{dt} \left(\frac{\partial \mathcal{J}}{\partial \dot{x}^i} \right) = \frac{d}{dt} \left(\frac{\partial \mathcal{J}}{\partial \dot{x}^i} \right) - \frac{\partial \mathcal{J}}{\partial x^i} = F^i, \quad i = 1, \dots, 3N, \text{ the}$$

Lagrange equations in euclidean space; we then pulled back by φ to get a

system of equations in the same form with respect to the q^i . What matters

here is that the Lagrange equations in fact remain invariant under pullback

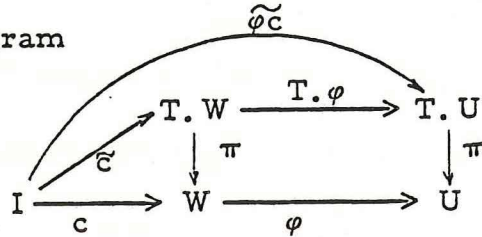
by any 1-1 map φ . This will be done in the next section.

§6. Lifting Paths

Let W be a differentiable manifold, and $c: I \rightarrow W$ be a path in W .

We define the lifted path $\tilde{c}: I \rightarrow T.(W)$ in the tangent bundle $T.(W)$ by

$\tilde{c}(t) = (c(t), \tau_{c(t)}c)$. Let $\varphi: W \rightarrow U$. Then $(T.\varphi)(\tilde{c}) = (\tilde{\varphi}c)$, since by definition $\tilde{\varphi}c(t) = (\varphi c(t), \tau_{\varphi c(t)}(\varphi c))$ and $(T.\varphi)(\tilde{c}(t)) = (\varphi c(t), T_{ct}\varphi(\tau_{ct}c))$ by definition of $T.\varphi$. We indicate the relationships of the above functions by a commutative diagram



If we choose coordinates $r^1, \dots, r^m: W \rightarrow \mathbb{R}$ for W and $q^1, \dots, q^n: U \rightarrow \mathbb{R}$ for U , we have coordinates $\dot{r}^1, \dots, \dot{r}^m; r^1\pi, \dots, r^m\pi: T.W \rightarrow \mathbb{R}$ for the tangent bundle of W and $\dot{q}^1, \dots, \dot{q}^n; q^1\pi, \dots, q^n\pi: T.U \rightarrow \mathbb{R}$ as coordinates for $T.U$.

The Jacobian $J = J(\varphi)$ of φ is the $n \times m$ matrix defined by $J_j^i = \frac{\partial q^i}{\partial r^j} \varphi$ ($= \frac{\partial q^i}{\partial r^j}$ for short), where $i = 1, \dots, n$ and $j = 1, \dots, m$. The Jacobian of $T.\varphi$ is defined similarly. It is a $2n \times 2m$ matrix of the form shown below

	r^1	\dots	r^m	\dot{r}^1	\dots	\dot{r}^m
q^1	$J(\varphi)$			0		
\vdots						
q^n	?			$J(\varphi)$		
\dot{q}^1						
\vdots						
\dot{q}^n						

The lower right block can be calculated as follows

$$\frac{dq^i}{dt} = \sum_{j=1}^m \frac{\partial q^i}{\partial r^j} \frac{dr^j}{dt} ; \quad \dot{q}^i = \sum_{j=1}^m \frac{\partial q^i}{\partial r^j} \dot{r}^j .$$

Therefore

$$\frac{\partial \dot{q}^i}{\partial \dot{r}^j} = \frac{\partial q^i}{\partial r^j} \quad \text{for all } i \text{ and } j .$$

Thus the equations for the transformation $T \cdot \varphi$ in terms of coordinates are

$$\dot{q}^i = \sum_{j=1}^m \frac{\partial q^i}{\partial r^j} \dot{r}^j, \quad q^i = \varphi(r^1, \dots, r^m).$$

The invariant description is $T \cdot \varphi(b, \tau_b c) = (\varphi b, \tau_{\varphi b}(\varphi c))$.

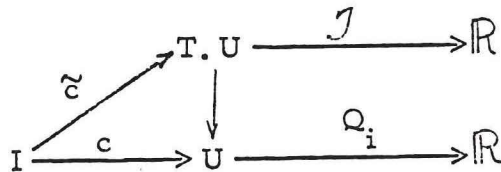
Given U , ω a 1-form on U , and $\mathcal{J}: T.U \rightarrow \mathbb{R}$ a smooth function, we define: a path c in U satisfies Lagrange's equation (with respect to \mathcal{J} and ω) in the coordinates q^1, \dots, q^n of U where

$$\omega = \sum_{j=1}^n Q_j dq^j, \quad \mathcal{J} = \mathcal{J}(q^1, \dots, q^n; \dot{q}^1, \dots, \dot{q}^n)$$

if

$$(1) \quad \frac{d}{dt} \left[\left(\frac{\partial \mathcal{J}}{\partial \dot{q}^i} \right) (\tilde{c}) \right] - \left(\frac{\partial \mathcal{J}}{\partial q^i} \right) (\tilde{c}) = Q_i c, \quad i = 1, \dots, n.$$

The functions are indicated in the diagram:



As a special case, suppose the forces are conservative. By definition, the 1-form ω (the work) is conservative if and only if there exists a smooth function $\mathcal{V}: U \rightarrow \mathbb{R}$ with $\omega = -d\mathcal{V}$, in coordinates

$$\sum_{i=1}^n Q_i dq^i = - \sum_{i=1}^n \frac{\partial \mathcal{V}}{\partial q^i} dq^i \quad \text{so} \quad Q_i = \frac{-\partial \mathcal{V}}{\partial q^i}. \quad \text{Thus (1) becomes}$$

$$\frac{d}{dt} \left[\left(\frac{\partial L}{\partial \dot{q}^i} \right) \tilde{c} \right] - \left(\frac{\partial L}{\partial q^i} \right) \tilde{c} = 0, \quad \text{where } L \text{ is defined by}$$

$$L = (\mathcal{J} - \mathcal{V} \cdot \pi): T.U \rightarrow \mathbb{R}$$

