

"smooth"; to be precise about this, we regard X as a map of U into $T(U)$. Since $Xa \in T_a(U)$, we can write X as $a \rightsquigarrow (a, Xa)$.

Definition. A vector field is a smooth function $X: U \rightarrow T(U)$ such that the composite $\pi \circ X$ is the identity function: $\pi \circ X = 1_U$.

The last requirement means simply that

$$X: u \rightsquigarrow (u, \text{some tangent vector at } u).$$

Recall that the vector space $T^a U$ has the basis $d_a q^1, \dots, d_a q^n$, where the q^i are coordinates in U . It seems natural to put coordinates on $T(U)$ so that the point $(a, w) \in T(U)$ is viewed as $(q^1, \dots, q^n, h^1 d_a q^1, \dots, h^n d_a q^n)$, where the q^i and the h^i are the coordinates of a and w , respectively. Thus $w = h^1 d_a q^1 + \dots + h^n d_a q^n$. In a similar way, we put $2n$ coordinates on the tangent bundle. We have seen that, if f is a function on U , then

$$d_a f = \sum \left(\frac{\partial f}{\partial q^i} \right)_a d_a q^i$$

is a point of $T^a U$. Hence we can write

$$(*) \quad df = \sum \left(\frac{\partial f}{\partial q^i} \right) dq^i$$

for the function which assigns to every point a of U the cotangent vector $d_a f$; it is thus the cotangent-space analogue of a vector field. Expressions like the above constantly pop up in physics; for example, if the three components of force in space are represented by the functions F_i of x, y , and z , the infinitesimal work is usually defined by

$$dW = F_1 dx + F_2 dy + F_3 dz.$$

When we interpret dW, dx , and so forth, not as infinitesimals, but as cotangent vector fields, this equation resembles equation (*). This will actually be the way we put the notion of "infinitesimal" on a sound mathematical basis.

§4 Vector Bundles

Take a long narrow strip of paper, draw a line down the middle lengthwise, and paste the ends together by twisting once to form a figure called the möbius strip. The line down the middle now becomes a circle, and the surface can be thought of as composed of vectors ("fibers") perpendicular to the circle and radiating from it. We are about to see how the mobius strip can be viewed as part of a vector bundle over the circle.

First we define the simpler notion of pre-bundle. Given two sets U and V , their cartesian product $U \times V$ is defined to be the set of all pairs (u, v) with $u \in U, v \in V$. Given sets G, E, U , and maps σ and π forming the following diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow \pi \\ G & \xrightarrow{\sigma} & U \end{array}$$

we define the pullback $G \times_U E$ as the universal object making the square

$$\begin{array}{ccc} G \times_U E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ G & \longrightarrow & U \end{array}$$

commutative; (universal objects and commutative squares will be defined in

