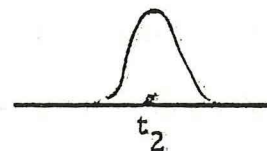


Lemma. If  $M: I \rightarrow \mathbb{R}$  is a smooth function and if

$$\int_{t_0}^{t_1} M \eta \, dt = 0$$

for all smooth functions  $\eta: I \rightarrow \mathbb{R}$  with  $\eta(t_0) = \eta(t_1) = 0$ , then  $M(t) = 0$  for all  $t$  with  $t_0 \leq t \leq t_1$ .

Proof. Suppose instead that  $M(t) \neq 0$  for some  $t = t_2$ , say that  $M(t_2) > 0$ . Then  $M(t) > 0$  on some small interval about  $t_2$  and we can choose a "bump" function  $b: I \rightarrow \mathbb{R}$  which is smooth, zero outside the interval, positive inside this interval, and 1 at  $t_2$ :



Then choosing  $\eta = bM$  as variation, the hypothesis gives

$$\int_{t_0}^{t_1} M \eta \, dt = \int_{t_0}^{t_1} bM^2 \, dt > 0.$$

This contradiction gives  $M = 0$ , as desired.

The methods used here are those of the Calculus of Variations. The result can be formulated more generally, as follows

Given  $h: T \times U \times I \rightarrow \mathbb{R}$ , consider paths  $c_0: I \rightarrow U$  which make  $\int_{t_0}^{t_1} h \tilde{c} \, dt$  stationary in comparison with other paths  $c$ ,  $c(t_0) = c_0(t_0)$ ,  $c(t_1) = c_0(t_1)$ . A necessary condition for this is Euler's equation:

$$\frac{d}{dt} \left( \frac{\partial h}{\partial \dot{q}^i} \right) = \frac{\partial h}{\partial q^i}, \quad i = 1, \dots, n.$$

In the special case when  $h$  is the Lagrangian function  $L_1$ , Euler's equations

are Lagrange's equations. In more general treatments, the smooth paths used above can be replaced by "piecewise" smooth paths.

§ 8 Bilinear and Quadratic Forms

The kinetic energy  $\mathcal{J}$  is usually a quadratic function of the velocities; that is,  $\mathcal{J} : T.U \rightarrow \mathbb{R}$  restricted to the fiber (tangent space) over a point of  $U$  is a quadratic function on that tangent space. We now study certain properties of such quadratic functions.

Let  $V$  be a finite dimensional vector space. Consider a function  $B : V \times V \rightarrow \mathbb{R}$  ( $(v, w) \rightsquigarrow B(v, w)$ ). We define  $B$  to be bilinear if  $B(v, w)$  is linear in  $v$  (with  $w$  fixed) and linear in  $w$  (with  $v$  fixed).

We define  $Q : V \rightarrow \mathbb{R}$  to be quadratic when,

$$1^\circ \quad Q(-v) = Q(v)$$

$$2^\circ \quad Q(u+v) - Q(u) - Q(v) \stackrel{\text{def.}}{=} 2Q^b(u, v) \text{ is bilinear in } u \text{ and } v.$$

That is,  $Q$  determines a symmetric bilinear function  $Q^b$ .

As a consequence of bilinearity, we have

$$Q(u+v+w) - Q(u) - Q(v+w) = Q(u+v) - Q(u) - Q(v) + Q(u+w) - Q(u) - Q(w).$$

$$\text{Letting } u = v = -w, \quad Q(u) - Q(u) = Q(2u) - Q(u) - Q(u) + 0 - Q(u) - Q(u)$$

and thus  $Q(2u) = 4Q(u)$ . (We must have  $Q(0) = 0$  since  $Q^b(0, 0) = 0$ .)

The assignment  $Q \rightsquigarrow_{\text{quadratic}} Q^b \stackrel{\text{def.}}{=} \frac{1}{2} [Q(u+v) - Q(u) - Q(v)]$  symmetric and bilinear, has an inverse  $B \rightsquigarrow B^\#$ . Define  $B^\#(u) = B(u, u)$ , for  $B$  symmetric and bilinear. Clearly,  $(Q^b)^\# = Q$ , since

$$(Q^b)^\#(u) = Q^b(u, u) = \frac{1}{2} [Q(u+u) - Q(u) - Q(u)] = \frac{1}{2} [4Q(u) - 2Q(u)] = Q(u).$$

Conversely,  $B^\#$  is quadratic:

$$1^\circ B^\#(-u) = B(-u, -u) = B(u, u)$$

and

$$2^\circ B^\#(u+v) - B^\#(u) - B^\#(v) \stackrel{\text{def}}{=} 2(B^\#)^\flat(u, v) = B(u+v, u+v) - B(u, u) - B(v, v) \\ = B(u, v) + B(v, u) = 2B(u, v).$$

And from this calculation, clearly  $(B^\#)^\flat = B$ .

Given  $Q: V \rightarrow \mathbb{R}$  quadratic and  $W \xrightarrow{\varphi} V$  a linear transformation, then  $Q\varphi: W \rightarrow \mathbb{R}$  is quadratic. For the proof, note

$$Q = V \xrightarrow{\Delta} V \times V \xrightarrow{B} \mathbb{R} \\ v \rightsquigarrow (v, v) \rightsquigarrow B(v, v) = Q(v)$$

where  $B = Q^\flat$  is symmetric and bilinear. Check that

$W \times W \xrightarrow{\varphi \times \varphi} V \times V \xrightarrow{B} \mathbb{R}$  is symmetric and bilinear. The rest of the proof is indicated by the commutative diagram:

$$\begin{array}{ccccc} W & \xrightarrow{\varphi} & V & & \\ \Delta \downarrow & & \Delta \downarrow & \searrow Q & \\ W \times W & \xrightarrow{\varphi \times \varphi} & V \times V & \xrightarrow{B} & \mathbb{R} \end{array} .$$

Choosing a basis  $e_1, \dots, e_n$  for  $V$ , and letting  $v = \sum_{i=1}^n q^i e_i$ , we have  $B(\sum_i q^i e_i, \sum_j q^j e_j) = \sum_{i,j} q^i q^j B(e_i, e_j)$ . Defining  $g_{ij} = B(e_i, e_j)$ , we have  $Q(v) = \sum_{i,j=1}^n g_{ij} q^i q^j$ ; we may call  $\|g_{ij}\|$  the matrix of  $Q$  for the basis  $e_i$ .

Consider the functions indicated in the following diagram

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{q^1} & T \cdot U & \xrightarrow{J} & \mathbb{R} \\ & \xleftarrow{q^n} & & \xrightarrow{L} & \\ & & \downarrow \pi & & \\ \mathbb{R} & \xleftarrow{q^1} & U & \xrightarrow{\gamma} & \mathbb{R} \\ & \xleftarrow{q^n} & & & \end{array}$$

