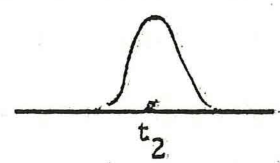
<u>Lemma.</u> If $M: I \longrightarrow \mathbb{R}$ is a smooth function and if

$$\int_{t_o}^{t_1} M\eta \, dt = 0$$

for all smooth functions $\eta: I \longrightarrow \mathbb{R}$ with $\eta(t_o) = \eta(t_1) = 0$, then $M(t) = 0$ for all t with $t_o \leq t \leq t_1$.

<u>Proof.</u> Suppose instead that $M(t) \neq 0$ for some $t = t_2$, say that $M(t_2) > 0$. Then $M(t) > 0$ on some small interval about t_2 and we can choose a "bump" function $b: I \longrightarrow \mathbb{R}$ which is smooth, zero outside the interval, positive inside this interval, and 1 at t_2:



Then choosing $\eta = bM$ as variation, the hypothesis gives

$$\int_{t_o}^{t_1} M\eta \, dt = \int_{t_o}^{t_1} bM^2 \, dt > 0.$$

This contradiction gives $M = 0$, as desired.

The methods used here are those of the Calculus of Variations. The result can be formulated more generally, as follows

Given $h: T.U \times I \longrightarrow \mathbb{R}$, consider paths $c_o: I \longrightarrow U$ which make $\int_{t_o}^{t_1} h\tilde{c} \, dt$ stationary in comparison with other paths c, $c(t_o) = c_o(t_o)$, $c(t_1) = c_o(t_1)$. A necessary condition for this is Euler's equation:

$$\frac{d}{dt}\left(\frac{\partial h}{\partial \dot{q}^i}\right) = \frac{\partial h}{\partial q^i}, \quad i = 1, \ldots, n.$$

In the special case when h is the Lagrangian function L_1, Euler's equations are Lagrange's equations. In more general treatments, the smooth paths used above can be replaced by "piecewise" smooth paths.

§ 8 Bilinear and Quadratic Forms

The kinetic energy $\mathcal{J}$ is usually a quadratic function of the velocities; that is, $\mathcal{J} : T.U \longrightarrow \mathbb{R}$ restricted to the fiber (tangent space) over a point of U is a quadratic function on that tangent space. We now study certain properties of such quadratic functions.

Let V be a finite dimensional vector space. Consider a function $B: V \times V \longrightarrow \mathbb{R}$ $((v, w) \rightsquigarrow B(v, w))$. We define B to be bilinear if $B(v, w)$ is linear in v (with w fixed) and linear in w (with v fixed). We define $Q: V \longrightarrow \mathbb{R}$ to be <u>quadratic</u> when,

1° $Q(-v) = Q(v)$

2° $Q(u+v) - Q(u) - Q(v) \overset{\text{def.}}{=\!=} 2Q^\flat(u, v)$ is bilinear in u and v.

That is, Q determines a symmetric bilinear function $Q^\flat$.

As a consequence of bilinearity, we have

$$Q(u+v+w) - Q(u) - Q(v+w) = Q(u+v) - Q(u) - Q(v) + Q(u+w) - Q(u) - Q(w).$$

Letting $u = v = -w$, $Q(u) - Q(u) = Q(2u) - Q(u) - Q(u) + 0 - Q(u) - Q(u)$ and thus $Q(2u) = 4Q(u)$. (We must have $Q(0) = 0$ since $Q^\flat(0,0) = 0$.)

The assignment $Q \underset{\text{quadratic}}{\rightsquigarrow} Q^\flat \overset{\text{def.}}{=\!=} \frac{1}{2}[Q(u+v) - Q(u) - Q(v)]$ symmetric and bilinear, has an inverse $B \rightsquigarrow B^\#$. Define $B^\#(u) = B(u, u)$, for B symmetric and bilinear. Clearly, $(Q^\flat)^\# = Q$, since

$$(Q^{\flat})^{\#}(u) = Q^{\flat}(u,u) = \frac{1}{2}[Q(u+u) - Q(u) - Q(u)] = \frac{1}{2}[4Q(u) - 2Q(u)] = Q(u).$$

Conversely, $B^{\#}$ is quadratic:

$$1^{\circ} \quad B^{\#}(-u) = B(-u,-u) = B(u,u)$$

and

$$2^{\circ} \quad B^{\#}(u+v) - B^{\#}(u) - B^{\#}(v) \quad (\stackrel{\text{def}}{=\!=} 2(B^{\#})^{\flat}(u,v)) = B(u+v, u+v) - B(u,u) - B(v,v)$$

$$= B(u,v) + B(v,u) = 2B(u,v).$$

And from this calculation, clearly $(B^{\#})^{\flat} = B$.

Given $Q: V \longrightarrow \mathbb{R}$ quadratic and $W \xrightarrow{\varphi} V$ a linear transformation, then $Q\varphi: W \longrightarrow \mathbb{R}$ is quadratic. For the proof, note

$$Q = V \xrightarrow{\Delta} V \times V \xrightarrow{B} \mathbb{R}$$
$$v \rightsquigarrow (v,v) \rightsquigarrow B(v,v) = Q(v)$$

where $B = Q^{\flat}$ is symmetric and bilinear. Check that $W \times W \xrightarrow{\varphi \times \varphi} V \times V \xrightarrow{B} \mathbb{R}$ is symmetric and bilinear. The rest of the proof is indicated by the commutative diagram:

$$\begin{tikzcd}
W \arrow[r, "\varphi"] \arrow[d, "\Delta"'] & V \arrow[d, "\Delta"'] \arrow[dr, "Q"] & \\
W \times W \arrow[r, "\varphi \times \varphi"] & V \times V \arrow[r, "B"] & \mathbb{R}
\end{tikzcd}$$

.

Choosing a basis $e_1, \ldots, e_n$ for V, and letting $v = \sum_{i=1}^{n} q^i e_i$, we have $B(\sum_i q^i e_i, \sum q^j e_j) = \sum_{i,j} q^i q^j B(e_i, e_j)$. Defining $g_{ij} = B(e_i, e_j)$, we have $Q(v) = \sum_{i,j=1}^{n} g_{ij} q^i q^j$; we may call $\| g_{ij} \|$ the matrix of Q for the basis e_i.

Consider the functions indicated in the following diagram

$$\begin{tikzcd}
\mathbb{R} & T.U \arrow[l, shift right, "\dot{q}^1"'] \arrow[l, shift left, "\dot{q}^n"] \arrow[r, shift left, "\mathcal{J}"] \arrow[r, shift right, "L"'] \arrow[d, "\pi"] & \mathbb{R} \\
\mathbb{R} & U \arrow[l, shift right, "q^1"'] \arrow[l, shift left, "q^n"] \arrow[r, "\mathcal{V}"] & \mathbb{R}
\end{tikzcd}$$

The $\dot{q}^i$ and q^i are the usual coordinates and $\mathcal{T}$ is quadratic and positive definite on each fiber, thus $\mathcal{T}$ is a Riemann metric on U. $L \underline{\underline{\text{def}}}\ \mathcal{T} - \mathcal{V}\pi$ is a quadratic plus a constant on each fiber. Let $\mathcal{T} = \sum_{i,j=1}^{n} g_{ij}\dot{q}^i\dot{q}^j$ where $g_{ij}: U \longrightarrow \mathbb{R}$, that is for $u \in U$, $(g_{ij}(u))$ is the positive definite symmetric matrix associated with the quadratic form $\mathcal{T}$ induces on $T_u U$, the fiber over U. But we have already noted that such a Riemann metric gives an isomorphism

$$T_u U \longrightarrow (T_u U)^* = T^u U$$

of the tangent space to its dual, the cotangent space. This isomorphism carries a point with coordinates $(\dot{q}^1, \dots, \dot{q}^n)$ to the point with coordinate p_i, where

$$p_i = \sum g_{ij}\dot{q}^j = \frac{\partial \mathcal{T}}{\partial \dot{q}^i} .$$

If we apply this isomorphism to each fiber of the tangent bundle, we get a smooth map $\mathcal{L} : T_\cdot U \longrightarrow T^\cdot U$ called the Legendre transformation, as in the diagram

$$\begin{tikzcd}
\mathbb{R} & T_\cdot U \arrow[l, "{\dot q^1, \dots, \dot q^n}"'] \arrow[rr, "\mathcal{L}"] \arrow[dr, "\pi_\cdot"'] & & T^\cdot U \arrow[r, "{p^1, \dots, p^n}"] \arrow[dl, "\pi^\cdot"] & \mathbb{R} \\
& & U \arrow[r, "{q^1, \dots, q^n}"'] & \mathbb{R} &
\end{tikzcd}$$

The coordinates for $T_\cdot U$ are $q^1\pi_\cdot, \dots, q^n\pi_\cdot, \dot{q}_1, \dots, \dot{q}_n$ and for $T^\cdot U$ they are $q^1\pi^\cdot, \dots, q^n\pi^\cdot, p_1, \dots, p_n$. The map $\mathcal{L}$ is then defined by

$$q^i\pi^\cdot\mathcal{L} = q^i\pi_\cdot \quad , \quad p_i\mathcal{L} = \frac{\partial \mathcal{T}}{\partial \dot{q}^i} .$$

We now consider the properties of this Legendre transformation $\mathcal{L}$

§9 The Legendre Transformation

Suppose U is an open set in $\mathbb{R}^n$ with coordinates $q^1,\dots,q^n$ and we are given a smooth kinetic energy function $\mathcal{J}$ from $T_.U$ to $\mathbb{R}$. We shall assume $\mathcal{J}$ quadratic; i.e., $\mathcal{J} = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial \mathcal{J}}{\partial \dot{q}^i}\,\dot{q}^i$. Such a function determines a smooth function $\mathcal{L}$ defined by

$$\begin{tikzcd} \mathbb{R} & T_.U \arrow[l, "\mathcal{J}"'] \arrow[r, "\mathcal{L}"] \arrow[dr, "\pi_."'] & T^{\cdot}U \arrow[ll, bend right, "\hat{\mathcal{J}}"'] \arrow[d, "\pi^{\cdot}"] \\ & & U \end{tikzcd}$$

$p_i\mathcal{L} = \frac{\partial \mathcal{J}}{\partial \dot{q}^i}$, $q^i\pi^{\cdot}\mathcal{L} = q^i\pi_.$. We set $\hat{\mathcal{J}} = \mathcal{J}\mathcal{L}^{-1}$. From the quadratic assumption on $\mathcal{J}$ we have

$$2d\mathcal{J} = \sum d\Big(\frac{\partial \mathcal{J}}{\partial \dot{q}^i}\Big)\dot{q}^i + \sum_{i=1}^{n}\frac{\partial \mathcal{J}}{\partial \dot{q}^i}\,d\dot{q}^i .$$

Subtract $d\mathcal{J} = \sum_{i=1}^{n}\frac{\partial \mathcal{J}}{\partial q^i\pi_.}\,dq^i\pi_. + \sum_{i=1}^{n}\frac{\partial \mathcal{J}}{\partial \dot{q}^i}\,d\dot{q}^i$ to get

$d\mathcal{J} = \sum_{i=1}^{n} d\Big(\frac{\partial \mathcal{J}}{\partial \dot{q}^i}\Big)\dot{q}^i - \sum_{i=1}^{n}\frac{\partial \mathcal{J}}{\partial q^i\pi_.}\,dq^i\pi_.$ Then

$$\begin{aligned} d\hat{\mathcal{J}} &= d(\mathcal{J}\mathcal{L}^{-1}) = (\mathcal{L}^{-1})^{*}\,d\mathcal{J} \\ &= \sum_{i=1}^{n} d\Big(\frac{\partial \mathcal{J}}{\partial \dot{q}^i}\mathcal{L}^{-1}\Big)\dot{q}^i\mathcal{L}^{-1} - \sum_{i=1}^{n}\frac{\partial \mathcal{J}}{\partial q^i\pi_.}\mathcal{L}^{-1}\,d(q^i\pi_.\mathcal{L}^{-1}) . \end{aligned}$$

(Recall that if $X \xrightarrow{\phi} Y \xrightarrow{f} \mathbb{R}$ are smooth functions, then f pulls back via ϕ to the function $f\phi$ and we have $d(f\phi) = \phi^{*}df$, $\phi^{*}(g\,df) = (g\phi)\,\phi^{*}df$.)

Finally, using the defining equations for $\mathcal{L}$,

$$d\hat{\mathcal{J}} = \sum_{i=1}^{n}(dp_i)\dot{q}^i\mathcal{L}^{-1} - \sum_{i=1}^{n}\Big(\frac{\partial \mathcal{J}}{\partial q^i\pi_.}\mathcal{L}^{-1}\Big)dq^i\pi^{\cdot} .$$

But also $d\hat{\mathcal{J}} = \sum_{i=1}^{n} \frac{\partial \hat{\mathcal{J}}}{\partial p_i} dp_i + \sum_{i=1}^{n} \frac{\partial \hat{\mathcal{J}}}{\partial q^i \pi^\bullet} dq^i \pi^\bullet$, and the coefficients of the differentials are unique. Thus

$$\dot{q}^i \zeta^{-1} = \frac{\partial \hat{\mathcal{J}}}{\partial p_i} \quad \text{and} \quad -\frac{\partial \mathcal{J}}{\partial q^i \pi_\bullet} \zeta^{-1} = \frac{\partial \hat{\mathcal{J}}}{\partial q^i \pi^\bullet},$$

which are the equations of the transferred kinetic energy $\hat{\mathcal{J}}$ in terms of the given one $\mathcal{J}$.

Now if we have a mechanical system with Lagrangian $L = \mathcal{J} - \mathcal{V}$ in which a path c satisfies Lagrange's equations, then these last equations yield Hamilton's equations for c. Let's follow the convention in mechanics of not writing in the maps $\pi_\bullet, \pi^\bullet$ and ζ, so the last equations are

$$\frac{dq^i}{dt} = \frac{\partial \hat{\mathcal{J}}}{\partial p_i} \quad \text{and} \quad -\frac{\partial \mathcal{J}}{\partial q^i} = \frac{\partial \hat{\mathcal{J}}}{\partial q^i}.$$

By definition of ζ

$$\frac{dp_i}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{J}}{\partial \dot{q}^i}\right) = \frac{d}{dt}\left(\frac{\partial \zeta}{\partial \dot{q}^i}\right)$$

$$= \frac{\partial \zeta}{\partial q^i} = \frac{\partial \mathcal{J}}{\partial q^i} - \frac{\partial \mathcal{V}}{\partial q^i} = -\left(\frac{\partial \hat{\mathcal{J}}}{\partial q^i} + \frac{\partial \mathcal{V}}{\partial q^i}\right).$$

So by setting $\mathcal{H} = \hat{\mathcal{J}} + \mathcal{V}\zeta^{-1}$, we have Hamilton's equations

$$\frac{dq^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q^i}.$$

<u>Exercise:</u> Let $n = 1$ and $\mathcal{J} = g\dot{q}^2$, where $U \xrightarrow{g} \mathbb{R}$. Calculate $\hat{\mathcal{J}}$.

We want to understand better how this scheme produced these equations. Notice first that χ maps T_aU into T^aU -- which is to say $(\chi, 1_U): \pi_\circ \longrightarrow \pi^\circ$ is a morphism of prebundles. Thus we may as well look at χ on each fibre of $T_\cdot U$ and paste the fibres together where we must. So consider a finite dimensional vector space V (think of this as T_aU for some $a \in U$):

a) At each point $v \in V$, $T_vV \cong V$ $(\tau_v c \rightsquigarrow w)$, where $c(t) = v+tw$ is a curve $I \longrightarrow V$. (Identify V with T_vV by this isomorphism.)

b) At each $v \in V$, $T^vV \cong V^*$ $(d_vf \rightsquigarrow \overline{d_vf}\,)$ where

$$\overline{d_vf}(w) = \frac{d}{dt}(f(v+tw))\Big|_{t=0} \quad \text{and} \quad V \xrightarrow{f} \mathbb{R} .$$

(Again identify V^* and T^vV.)

c) $T^\cdot V$ can be identified with $V \times V^*$ via

$$(v, d_vf) \rightsquigarrow (v, \overline{d_vf})$$

$$\begin{tikzcd} T^\cdot V \arrow[r] \arrow[d, "\pi^\cdot"'] & V \times V^* \arrow[d, "\pi"] \\ V \arrow[r, "="] & V \end{tikzcd} \qquad \text{where } \pi(v,\alpha) = v.$$

We have another projection $V \times V^* \longrightarrow V^*$ $(v,\alpha) \rightsquigarrow \alpha$.

d) A 1-form ω on V is a smooth function $\omega: T$— $T^\cdot V$ such that $\pi \circ \omega = 1_V$. Composition of σ with a 1-form ω produces a smooth function

$$\chi_\omega = \sigma \circ \omega .$$

$$\begin{tikzcd} V \times V^* \arrow[r, "\sigma"] \arrow[d, shift left, "\pi"] & V \\ V \arrow[u, shift left, "\omega"] \arrow[ur, "\chi_\omega"'] & \end{tikzcd}$$

e) In particular, suppose $L: V \longrightarrow \mathbb{R}$ is smooth (think of L as a Lagrangian function restricted to one fiber of $T.U$). Then dL is a 1-form and so determines $V \xrightarrow{\mathcal{L}_{dL}} V^*$. To compare this with the example at the beginning of this section, let the potential energy be zero so $L = \mathcal{J}$. The first defining equation for $\mathcal{L}$ says $\mathcal{L}_{d\mathcal{J}} = \sigma \circ d\mathcal{J}$ when written in coordinates. Returning to the general case, the explicit formula for $\mathcal{L}_{dL}$ gives each value $\mathcal{L}_{dL}V$ as a function of w:

$$\mathcal{L}_{dL}V(w) = \frac{d}{dt} L(v+tw)\Big|_{t=0} .$$

f) Let $e_1, \dots, e_n$ be a basis for V with coordinates $e^1, \dots, e^n$; then $e^1, \dots, e^n$ are a dual basis for V^* with coordinates $e_1, \dots, e_n$. The formula for $\mathcal{L}_{dL} = \mathcal{L}$ in these bases is

$$\mathcal{L}v(w) = \sum_{j=1}^{n} \left(\frac{\partial L}{\partial e^j} \Big|_v \right) e^j w .$$

Apply e_i to both sides and use $e_i e^j = \delta^i_j$ to get $e_i \mathcal{L} v = \frac{\partial L}{\partial e^i}\Big|_v$ or, as functions

$$e_i \mathcal{L} = \frac{\partial L}{\partial e^i} \quad , \quad i = 1, \dots, n$$

In the mechanical rotation $e^i = \dot{q}^i$ and $e_i = p_i$, since $V = T_a U$ and $V^* = T^a U$ for $a \in U$, U open in $\mathbb{R}^n$, and the result reads

$$p_i \mathcal{L} = \frac{\partial L}{\partial \dot{q}^i}$$

as before.

The Hamiltonian function arises from asking the question: when is $\mathcal{L}$ invertible? (This is probably not the way Hamilton found it.)

By the Inverse Function Theorem (see Abraham p. 10), $\mathcal{L}^{-1}$ exists if and only if the matrix $\left(\frac{\partial^2 L}{\partial e^i \partial e^j}\Big|_v \right)$ is non-singular at every $v \in V$.

g) Suppose $\mathcal{L}$ invertible. Does $\mathcal{L}^{-1}$ come from a smooth function on V^* in the same manner $\mathcal{L}$ came from a smooth L on V? That is, is there $V^* \xrightarrow{H} \mathbb{R}$ such that $\mathcal{L}^{-1} = \pi \circ dH$?

$$\begin{tikzcd}
V \times V^* \arrow[r, "\pi"] \arrow[d, shift left, "\sigma"] & V \arrow[dl, "\mathcal{L}^{-1}"] \\
V^* \arrow[u, shift left, "dH"] &
\end{tikzcd}$$

From part f) we have

$$dL = \sum_{i=1}^{n} (e_i\mathcal{L})de^i.$$

Let's try the formula dual to this one:

We want H so that $dH = \sum_{i=1}^{n} (e^i\mathcal{L}^{-1}) de_i$. Use the derivation property of d:

$$\sum_{i=1}^{n} (e^i\mathcal{L}^{-1})de_i = d[\sum_{i=1}^{n} (e^i\mathcal{L}^{-1})e_i] - \sum_{i=1}^{n} e_i d(e^i\mathcal{L}^{-1}).$$

The second term on the right is

$$- \sum_{i=1}^{n} e_i\mathcal{L}\mathcal{L}^{-1}\, d(e^i\mathcal{L}^{-1}) = -(\mathcal{L}^{-1})^* \sum_{i=1}^{n} (e_i\mathcal{L})de^i$$

$$= -(\mathcal{L}^{-1})^* dL = -d(L\mathcal{L}^{-1}).$$

Substitute this in our conjectured dH to get

$$dH = d[\sum_{i=1}^{n} (e^i\mathcal{L}^{-1})e_i - L\mathcal{L}^{-1}],$$

so H should be

$$H = \sum_{i=1}^{n} (e^i\mathcal{L}^{-1})e_i - L\mathcal{L}^{-1}.$$

The steps reverse so this is indeed the right formula. On elements,

$$H_y = \mathcal{L}_y^{-1} \cdot y - L\mathcal{L}_y^{-1} \quad \text{for } y \in V^*$$

usual inner product

and in mechanical notation

$$H = \sum_{i=1}^{n} \dot{q}^i p_i - L.$$

(Notice that we are leaving out $\mathcal{L}$ and $\mathcal{L}^{-1}$ as is customary in mechanics.) If, in particular, $L = \mathcal{J} - \mathcal{V}$ where $\mathcal{V}$ is a function of only the q^i's and $\mathcal{J}$ is quadratic as in our example, then

$$H = \sum_{i=1}^{n} \dot{q}^i \frac{\partial \mathcal{J}}{\partial \dot{q}^i} - L = 2\mathcal{J} - L = \mathcal{J} + \mathcal{V}$$

Given the Lagrangian L, define the <u>Action</u> $A: V \longrightarrow \mathbb{R}$, and <u>Energy</u> $E: V \longrightarrow \mathbb{R}$ by $Av = <\mathcal{L}_{dL}v, v>$, $Ev = <\mathcal{L}_{dL}v, v> - Lv$. Both are smooth functions.

The last lecture proved most of the following theorem:

<u>Theorem</u> (Legendre): If V is a finite dimensional real vector space of dimension n, then

1) for each $v \in V$, there is a natural isomorphism $T^vV \cong V^*$ (which we consider equality below)

2) for each smooth $L: V \longrightarrow \mathbb{R}$, there is a smooth

$$V \xrightarrow{\mathcal{L}_{dL}} V^* \qquad (v \rightsquigarrow d_vL)$$

In coordinates

$$dL = \sum_{i=1}^{n} (e_i \mathcal{L}_{dL}) de^i$$

with $e_1, \ldots, e_n$ a basis for V. $\mathcal{L}_{dL}$ is called the <u>Legendre transformation for L</u>.

3) the function $\mathcal{L} = \mathcal{L}_{dL}$ is invertible if and only if the matrix $\left(\frac{\partial^2 L}{\partial e^i \partial e^j}\Big|_v\right)$ is non-singular at each $v \in V$. If this is so $\mathcal{L}^{-1}: V^* \longrightarrow V$ is the Legendre transformation for $H: V^* \longrightarrow \mathbb{R}$ defined by

$$Hy = \langle y, \mathcal{L}^{-1} y\rangle - L\mathcal{L}^{-1} y \quad = E\mathcal{L}^{-1} y .$$

4) in particular, if L is quadratic, then $\mathcal{L}$ is the isomorphism of V with its dual given by the inner product induced by L and $E \doteq L$ in this case.

[See Sternberg pp. 150-153, Goldstein pp. 215-   , Abraham §17 .]

<u>Corollary 1.</u> If U open in $\mathbb{R}^n$, I open interval $\subset \mathbb{R}$ and $L: T.U \times I \longrightarrow \mathbb{R}$ is smooth, then L on each fibre determines

$$\begin{tikzcd} T.U \times I \arrow[rr, "\mathcal{L}"] \arrow[dr] & & T^{\cdot}U \times I \arrow[dl] \\ & U \times I & \end{tikzcd}$$

and all parts of the theorem hold for this $\mathcal{L}$. (Abraham calls $\mathcal{L}$ the fibre derivative of L.)

<u>Corollary 2.</u> Let c be a path in U, $\tilde{c}$ lifted path in $T.U$. If $\tilde{c}$ satisfies Lagrange's equations for L, then $\mathcal{L}\tilde{c}$ satisfies the canonical differential equations for H (Hamilton's equations):

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} .$$

<u>Exercise</u> Prove Corollary 2.

Forget the projection, so $T^{\cdot}U$ is an open set in $\mathbb{R}^{2n}$ and on it sits a first order differential equation -- Hamilton's equation of Corollary 2. We shall consider changes of coordinates in this 2n-dimensional space which leave this differential equation invariant. To do this we need to consider 2-forms, and in general k-forms.

Chapter II. Tensors and Exterior Forms

§10. Vector Fields

Let U be open in $\mathbb{R}^n$ with coordinates $q^1, \ldots, q^n$. A <u>vector field</u> X on U is a cross-section of the tangent bundle of U; i.e., $\pi_{\bullet} X(a) = a$ for all $a \in U$.

$$\begin{tikzcd} T_{\bullet}U \arrow[d, bend left, "\pi"] \\ U \arrow[u, bend left, "X"] \end{tikzcd}$$

For example, for given coordinates q^i, we can define a vector field D "along the axis q^i" by $D^i(a) = (a, \overline{D}^i a)$, where $\overline{D}^i(a) = \tau_a$ (path along i^{th} coordinate axis) = unit vector in i^{th} direction in $T_a U$, for $i = 1, \ldots, n$. This clearly defines a cross-section D^i of the tangent bundle. The set of vector fields on U is an $\mathcal{F}$-module, where $\mathcal{F}$ is the ring of smooth functions $U \longrightarrow \mathbb{R}$. The $\mathcal{F}$-module structure is given by the equations

$$(X_1 + X_2)a = X_1 a + X_2 a$$

$$fX_1(a) = f(a) \cdot X_1 a ,$$

for $a \in U$, $f \in \mathcal{F}$ and X_1, X_2 vector fields. The vectors $D^i(a)$, $i = 1, \ldots, n$ form a basis of $T_a U$, so

$$Xa = \sum_{i=1}^{n} (X_i a) D^i(a);$$

thus $X = \sum_{i=1}^{n} X_i D^i$, where the functions $X_i : U \longrightarrow \mathbb{R}$ are smooth and unique. This says that the vector fields $D^1, \ldots, D^n$ are a basis for the set of vector fields on U as a real vector space.

Each vector field X produces a function called the <u>Lie derivative</u> $\mathcal{F} \xrightarrow{L_X} \mathcal{F}$ with

$$L_X f(a) = \langle d_a f, Xa \rangle = \text{derivative of } f \text{ along } X.$$

Here we need "smooth" to mean C^∞, since otherwise $L_X f$ has one lower order of differentiability than f. The function L_X has the properties

(1) L_X is $\mathbb{R}$-linear,

(2) $L_X(f\cdot g) = f\cdot L_X g + g\cdot L_X f$.

Property (1) is a consequence of the linearity of d_a and $\langle\ ,Xa\rangle$. For (2)

$$\begin{aligned} L_X(f\cdot g)a &= \langle d_a f\cdot g, Xa\rangle \\ &= \langle f(a)\cdot d_a g + g(a)\cdot d_a f, Xa\rangle \\ &= f(a)\langle d_a g, Xa\rangle + g(a)\langle d_a f, Xa\rangle \\ &= (f\cdot L_X g + g\cdot L_X f)a . \end{aligned}$$

In coordinates, $X = \sum_{i=1}^{n} X_i D^i$ so

$$\begin{aligned} L_X f &= \left\langle \sum_{j=1}^{n} \frac{\partial f}{\partial q^i}\, dq^j \,,\ \sum_{i=1}^{n} X_i D^i \right\rangle \\ &= \sum_{i=1}^{n} X_i \sum_{j=1}^{n} \frac{\partial f}{\partial q^i} \langle dq^j, D^i\rangle \\ &= \sum_{i=1}^{n} X_i \frac{\partial f}{\partial q^i}\ , \quad \text{since } \langle dq^j, D^i\rangle = \delta_i^j . \end{aligned}$$

In particular, if $X = D^i$, then

$$L_{D^i} f = \frac{\partial f}{\partial q^i}\ ,$$

so L_{D^i} is sometimes written $\partial/\partial q^i$.

Definition. A derivation on the ring $\mathfrak{F}$ is an $\mathbb{R}$-linear function $\theta: \mathfrak{F} \longrightarrow \mathfrak{F}$ such that

$$\theta(f \cdot g) = f \cdot \theta g + g \cdot \theta f.$$

Each L_X is a derivation on $\mathfrak{F}$; in fact, these are all the derivations on $\mathfrak{F}$.

Theorem. For every derivation θ on $\mathfrak{F}$, there is a unique vector field X such that $\theta = L_X$.

Proof. Take $a \in U$. Since translations are invertible functions which preserve all differentiable structures, we may as well assume $a = 0$. Since U is open, it contains a ball with center at the origin: for any u in that ball define a path c in U by

$$c(t) = tu.$$

The Fundamental Theorem of Calculus gives us the equation

$$fc(1) - fc(0) = \int_0^1 \frac{dfc}{dt}\, dt$$

$$= \int_0^1 \sum_{i=1}^{n} \frac{\partial fc}{\partial q^i}\, q^i(u)\, dt.$$

Set $h_i(u) = \int_0^1 \frac{\partial fc}{\partial q^i}\, dt$ and notice that $q^i(u)$ is independent of t:

$$f(u) = f(0) + \sum_{i=1}^{n} h_i(u) q^i(u).$$

Then, by the defining property of θ:

$$\theta f(0) = \sum_{i=1}^{n} h_i(0)\theta q^i(0) + q^i(0)\theta h_i(0)$$

$$= \sum_{i=1}^{n} \theta q^i(0) \left.\frac{\partial f}{\partial q^i}\right|_0 ,$$

since $q^i(0) = 0$ and $h_i(0) = \left.\dfrac{\partial f}{\partial q^i}\right|_0$. For any $f \in \mathfrak{F}$ then we have

$$\theta f = \sum_{i=1}^{n} \theta q^i \cdot \frac{\partial f}{\partial q^i} ,$$

which is exactly $L_X f$ when $X_i = \theta q^i$. The X_i uniquely determine X, so the theorem is proved.

§11. The Tensor Product

This section begins with the material in MacLane and Birkhoff Algebra, Chapter VI, §§4 and 5, and Chapter IX §§7 and 8.

A tensor is sometimes described by symbols with many indices, upper and lower. To really understand tensors, we must understand their relation to the basic vector space V under discussion. Tensors are in fact elements of new vector spaces built up out of V and its dual space by the operation of tensor product.

Given vector spaces V and W, a tensor product of V and W is a vector space, which we will write $V \otimes W$, together with a bilinear function $\otimes : V \times W \longrightarrow V \otimes W$, which have the following property: if $B: V \times W \longrightarrow U$ is any bilinear function, then there is a unique linear map $F: V \otimes W \longrightarrow U$ such that the diagram below commutes:

$$\begin{tikzcd} & V \otimes W \arrow[d, "F"] \\ V \times W \arrow[ur] \arrow[r, "B"'] & U \end{tikzcd}$$

Briefly, we say that $\otimes$ is "universal" among bilinear functions on $V \times W$. If we write the image of the pair $(v, w) \in V \times W$ under the map $\otimes$ as