<u>Definition.</u> An exterior k-form ω on U $(k = 0, 1, \ldots, n)$ is a smooth cross-section of

$$\begin{array}{c} \Lambda^k U \\ \downarrow \pi \\ U \end{array}.$$

Thus a k-exterior form ω on $\mathbf{U}$ is a smooth map $U \xrightarrow{\omega} \Lambda^k U$ of the form $a \rightsquigarrow (a, \tilde{\omega}a)$ where $\tilde{\omega}a \in \Lambda_k(T^a U)$.

Note that a 1-form by this definition coincides with our previous definition of a 1-form, and that a 0-form is just a smooth map from U to $\mathbb{R}$.

<u>Definition.</u> $\Omega^k(U)$ = the set of all k-forms ω on U.

We remark that

1) $\Omega^k(U)$ is a vector space over $\mathbb{R}$, with vector space opeations as follows: if $\omega \in \Omega^k(U)$, $\alpha \in \mathbb{R}$ and ω: $a \rightsquigarrow (a, \tilde{\omega}a)$, then

$$\alpha\omega\colon a \rightsquigarrow (a, \alpha(\tilde{\omega}a)) \ ; \ \omega_1 + \omega_2 : a \rightsquigarrow (a. \tilde{\omega}_1 a + \tilde{\omega}_2 a).$$

2) $\Omega^k(U)$ is an $\mathcal{F}$-module, where $\mathcal{F}$ = the ring of smooth functions $f\colon U \longrightarrow \mathbb{R}$. The module action is given by $f\omega\colon a \rightsquigarrow (a, f(a)\tilde{\omega}a)$.

3) For each k and m, there is an exterior product

$$\Omega^k(U) \times \Omega^m(U) \longrightarrow \Omega^{k+m}(U)$$

$$(\omega, \eta) \rightsquigarrow \omega \wedge \eta$$

defined by

$$\omega \wedge \eta : a \rightsquigarrow (a, \tilde{\omega}(a) \wedge \tilde{\eta}(a)) .$$

This is well-defined since $\tilde{\omega}(a)$ and $\tilde{\eta}(a)$ are both elements of $\Lambda(T^a U)$. The exterior product is associative and bilinear. Thus the $\Omega^k(U)$ form a graded algebra $\Omega^*(U)$ over $\mathcal{F}$.

Since $\Lambda^k(U)$ has as basis all k-fold wedge products of the dq^i, where $q^1, \ldots, q^n$ are coordinates in U, any k-form ω is of the form

$$\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} f^{i_1 \ldots i_k} \wedge dq^{i_1} \wedge \ldots \wedge dq^{i_k},$$

where $f^{i_1 \ldots i_k} : U \longrightarrow \mathbb{R}$. If the $p^{i_1 \ldots i_k}$ are the coordinates corresponding to the $dq^{i_1} \wedge \ldots \wedge dq^{i_k}$, then

$$f^{i_1 \ldots i_k} = p^{i_1 \ldots i_k} \circ \omega .$$

Given a smooth map $U \xrightarrow{\varphi} U'$, we have an induced map $T^a U \xleftarrow{\varphi^*} T^a U'$, which in turn determines, via the properties of the exterior algebra, a map $\Omega^k U \xleftarrow{\varphi^*} \Omega^k U'$ such that φ^* is linear and

$$\varphi^*(\omega \wedge \eta) = (\varphi^* \omega) \wedge (\varphi^* \eta).$$

Then if $q^1, \ldots, q^n$ are coordinates in U and $r^1, \ldots, r^n$ in U', it will follow that

$$\varphi^*(dq^1 \wedge \ldots \wedge dq^n) = \det\left(\frac{\partial q^i \varphi}{\partial r^j}\right) dr^1 \wedge \ldots \wedge dr^n ,$$

which is just the usual change-of-coordinates rule.

Now we define the basic operations of exterior differentiation of forms.

<u>Theorem</u>. There exists a unique linear map

$$d : \Omega^k(U) \longrightarrow \Omega^{k+1}(U) \qquad \text{for each } k$$

such that

1) for $f \in \Omega^0$, df = usual differential of f

2) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$, where k = degree of ω [i.e.,

every time d is moved past something of degree 1, the sign changes]

3) $dd\omega = 0$ for all ω

(d is called the exterior derivative)

Proof. If d exists, then by linearity and (1)-(3),

$$d\left(\sum f^{i_1 \dots i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k}\right)$$
$$= \sum d\left(f^{i_1 \dots i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k}\right)$$
$$= \sum df^{i_1 \dots i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k} ,$$

since all other summands have a factor of the form ddq^i and hence are zero by (3).

Since every k-form ω can be written as

$$\omega = \sum f^{i_1 \dots i_k} dq^{i_1} \dots dq^{i_k}$$

for smooth functions $f^{i_1 \dots i_k}$, the above shows that d, if it exists, must be unique, since it is determined by its effect on 0-forms, which is specified by (1). But the above expression also defines a function $d: \Omega^k(U) \longrightarrow \Omega^{k+1}(U)$ for each k. It is easily verified that this d is linear and satisfies (1)-(3).

Chapter III. HAMILTONIAN MECHANICS

17. Calculus of Variations

Suppose that M is a local manifold with coordinates $y^1, \ldots, y^m$, and that

$$K: T.M \times I \longrightarrow \mathbb{R}$$

is a smooth function. We have already seen that the integral

$$\int_{t_0}^{t_1} K\, dt$$

is stationary along the path $c: I \longrightarrow M$ if and only if c satisfies Euler's Equations for K:

$$\frac{d}{dt} \frac{\partial K}{\partial \dot{y}^i} - \frac{\partial K}{\partial y^i} = 0 \quad , \quad i = 1, \ldots, m.$$

In particular, using $K = L$, we obtained Lagrange's equations in this way.

Theorem. (Generalized Hamilton's Principle): Given U with coordinates $q^1, \ldots, q^n$, additional coordinates $p_1, \ldots, p_n$ in $T^{\cdot} U$, and a smooth function $H: T^{\cdot} U \longrightarrow \mathbb{R}$, then Hamilton's equations for H are just the Euler equations for the function

$$K = \sum_{i=1}^{n} p_i \dot{q}^i - H,$$

where $M = T^{\cdot} U$.

Proof. $T^{\cdot} U$ has coordinates $q^1, \ldots, q^n$ and $p_1, \ldots, p_n$, so Euler's equations correspondingly take two forms:

1) for the q^i's

$$\frac{d}{dt} \frac{\partial K}{\partial \dot{q}^i} - \frac{\partial K}{\partial q^i} = 0$$

or

$$\frac{d}{dt} p_i = - \frac{\partial H}{\partial q^i} \quad ,$$

2) for the p_i's

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{p}_i} - \frac{\partial K}{\partial p_i} = 0$$

or

$$-\dot{q}^i + \frac{\partial H}{\partial p_i} = 0 \text{ , since } K \text{ is independent of } \dot{p}_i \text{ .}$$

In the above, we would like to express

$$\int \sum p_i \dot{q}^i \, dt$$

in terms of a differential form. Suppose

$$c : I \longrightarrow M$$

is a curve in M, lifted to

$$\tilde{c} : I \longrightarrow T.M.$$

Let ω be the 1-form

$$\omega = \sum_{i=1}^{n} p_i dq^i \quad \text{on } M.$$

Pulling ω back to I via c, we get

$$c^*\omega = \sum_{i=1}^{n} p_i c(c^* dq^i) = \sum_{i=1}^{n} p_i c \frac{dq^i c}{dt} \, dt \text{ .}$$

Thus what we mean by

$$\int_{t_o}^{t_1} \sum_{i=1}^{n} p_i \dot{q}^i \, dt$$

is just

$$\int_{t_o}^{t_1} (c^*\omega) \, dt \text{ .}$$

In what follows, we'll find it convenient to use the 2-form

$$\Omega = d\omega = \sum_{i=1}^{n} dp_i \wedge dq^i$$

rather than ω itself.

$M = T.U$ is called the <u>phase space</u> of the system, and U is called the <u>configuration space</u>.

<u>Definition.</u> Let M' and M be phase spaces with coordinates $Q^1, \ldots, Q^n; P_1, \ldots, P_n$ and $q^1, \ldots, q^n; p_1, \ldots, p_n$ respectively. A smooth, one-to-one mapping $M' \longrightarrow M$ is called a <u>canonical transformation</u>, or <u>contact transformation</u>, with respect to the given coordinates, if

$$\varphi^*\left(\sum_{i=1}^{n} dp_i \wedge dq^i\right) = \sum_{i=1}^{n} dP_i \wedge dQ^i .$$

The importance of these transformations is that they preserve Hamilton's equations:

<u>Theorem.</u> Given a canonical transformation $\varphi\colon M' \longrightarrow M$, a smooth map $H\colon M \longrightarrow \mathbb{R}$, and a path c in M' such that the image of c is contained in some open, simply connected subset of M'. If φc satisfies Hamilton's canonical equations in M for H, then c satisfies them in M' for $\varphi^* H$

$$\begin{tikzcd} \mathbb{R} \arrow[d, "c"'] \arrow[dr, dashed, "\varphi c"] & \\ M' \arrow[r, "\varphi"] \arrow[dr, dashed, "\varphi^* H"'] & M \arrow[d, "H"] \\ & \mathbb{R} \end{tikzcd}$$

<u>Proof.</u> We want to show that

$$\int_{t_o}^{t_1} \left(\sum_{i=1}^{n} P_i dQ^i - \varphi^* H\right) dt$$

is stationary over the path c with respect to nearby smooth paths with the same end-points.

Since the transformation φ is canonical

$$\begin{aligned} d\varphi\left(\sum p_i dq^i\right) &= \varphi^*\left(\sum dp_i \wedge dq^i\right) \\ &= \sum dP_i \wedge dQ^i = d\left(\sum P_i dQ^i\right), \end{aligned}$$

so the 1-form

$$\rho = \varphi^*\left(\sum p_i dq^i\right) - \sum P_i dQ^i$$

is closed, i.e., $d\rho = 0$. Thus (by Poincaré's lemma, to be proved below) there exists a 0-form F on the open, simply connected subset of M containing the image of c, such that

$$\rho = dF$$

on that subset. Then over any curve γ in this subset,

$$\begin{aligned}
&\int_{t_o}^{t_1} \left(\sum_{i=1}^{n} P_i dQ^i - \varphi^* H\right) dt \\
&\qquad = \int_{t_o}^{t_1} \left(\varphi^*\left(\sum p_i dq^i\right) - \rho - \varphi^* H\right) dt \\
&\qquad = \int_{t_o}^{t_1} \left[\varphi^*\left(\sum p_i dq^i - H\right) - dF\right] dt \\
&\qquad = \int_{t_o}^{t_1} \varphi^*\left(\sum p_i dq^i - H\right) dt - F(\gamma(t_1)) + F(\gamma(t_o)) \\
&\qquad = \int_{t_o}^{t_1} \left(\sum p_i dq^i - H\right) dt - F(\gamma(t_1)) + F(\gamma(t_o)),
\end{aligned}$$

where the last integration is over $\varphi\gamma$. But we are comparing c with nearby curves γ with the same endpoints. Thus $F(\gamma(t_1)) - F(\gamma(t_o))$ is a constant. Then since

$$\int_{t_o}^{t_1} \left(\sum p_i dq^i - H\right) dt$$

is stationary for the path φc, the above relation says that

$$\int_{t_o}^{t_1} \left(\sum P_i dQ^i - \varphi^* H\right) dt$$

is stationary over c.

Examples:

1) Let $M \xrightarrow{\varphi} M$ be the transformation defined by

$$p_i\varphi = -Q^i$$
$$q^i\varphi = P_i \, .$$

Then
$$\varphi^*\left(\sum_{i=1}^{n} dp_i \wedge dq^i\right) = \sum_{i=1}^{n} d(p_i\varphi)\wedge d(q^i\varphi)$$
$$= -\sum_{i=1}^{n} dQ^i \wedge dP_i$$
$$= \sum_{i=1}^{n} dP_i \wedge dQ^i \, .$$

Thus φ is a canonical transformation.

2) We will see later that if $V \xrightarrow{\varphi} U$ is a smooth, invertible map of configuration spaces, then the induced map $T^{\cdot}V \xleftarrow{\Phi} T^{\cdot}U$ of the cotangent spaces is a canonical transformation. Such a transformation is called a (canonical) point transformation.

§18. Application: The Harmonic Oscillator

a) The linear oscillator (cf. Goldstein, p. 24):

This concerns the following phase space:

U is one-dimensional with coordinate q,

$$T = \frac{1}{2} m \dot{q}^2 = \frac{1}{2m} p^2 ,$$
$$V = \frac{1}{2} k q^2 ,$$
$$H = T + V = \frac{1}{2}\left[\frac{p^2}{m} + kq^2\right].$$

We would like to find a canonical transformation of $T^{\cdot}U$ so that H is of a simpler form.

Let $\quad p = \sqrt{km}\ F \cos Q,$

$q = F \sin Q$, where F is a function of P.

Then

$$H = \frac{1}{2m}[p^2 + kmq^2] = \frac{k}{2}F^2 ,$$

$$dp \wedge dq = \sqrt{km}\,(F' \cos Q\, dP - F \sin Q\, dQ) \wedge (F' \sin Q\, dP + F \cos Q\, dQ)$$

$$= \sqrt{km}\ FF'\, dP \wedge dQ .$$

So our transformation will be canonical if

$$dP \wedge dQ = \sqrt{km}\ FF'\, dP \wedge dQ ,$$

i. e., if

$$\sqrt{km}\ FF' = 1.$$

We can integrate this to get

$$F = \frac{\sqrt{2p}}{(km)^{1/4}} .$$

Let $\omega = (km)^{1/2}$. Then

$$H = \frac{k}{2}F^2 = \omega P,$$

so Hamilton's equations become

$$\frac{dQ}{dt} = \omega \ , \quad \frac{dP}{dt} = 0 .$$

These may be immediately integrated as

$$Q = \omega t + \alpha \ , \qquad P = \text{constant},$$

and

$$p = m\omega\,(2p/m\omega)^{1/2} \cos(\omega t + \alpha),$$

$$q = (2p/m\omega)^{1/2} \sin(\omega t + \alpha).$$

These are of course the familiar equations for the (one-dimensional) harmonic oscillator.

b) As a further example of the technique of canonical transformations, we consider the case of any oscillation around an equilibrium position. A point of equilibrium is characterized, in configuration space, by the equations $\partial V/\partial q^i = 0$, all i. For n = 2, imagine the potential function V to be represented by a surface in space; at an equilibrium point this surface has a critical point, which in the stable case is a local minimum. The state of the system behaves like a marble rolling on the potential surface; it oscillates back and forth in the potential well. To see this, we expand T and V by Taylor series about the origin, neglecting all except the quadratic terms:

$$T = \sum a_{ij}\dot{q}^i\dot{q}^j \quad , \quad V = \sum b_{ij}q^iq^j \ ,$$

where the a_{ij} and b_{ij} are constants (here we've used the fact that the point at which $q^1 = q^2 = \ldots = q^n = 0$ is a critical point of V to eliminate the first-order terms in the Taylor expansion of V). We can find a linear transformation to new coordinates $\{r^i\}$ in which T is diagonal: $T = \sum (\dot{r}^i)^2$. The well-known principal-axis theorem now allows us to change coordinates again so that V also assumes a diagonal form, $V = \sum k_i(r^i)^2$; since these changes may be made by an orthogonal transformation (which preserves the inner product), T remains diagonal. But now that we have diagonalized both T and V, we see that each of the i coordinates r satisfies the equations $T = \dot{r}^2$, $V = kr^2$, which we have shown lead to simple harmonic motion. We say that small perturbations around a point of stable equilibrium produce simple harmonic motion in each suitably chosen coordinate.

§19. Canonical Transformations.

We will now prove that every point transformation (one that is given by a smooth, 1-1, onto map of configuration space) is a canonical transformation; at the same time we will be able to get a more natural invariant description of the basic form Ω, which we have been writing as $\Omega = \sum dp_i \wedge dq^i$. Recall that to each map $\varphi: U \to U'$ we associated a linear map $\varphi^*: T^{\varphi(a)}(U') \to T^a(U)$. In particular, we can regard $T^{\cdot}(U)$ as a local manifold, and the canonical projection π onto U as a smooth map of local manifolds. Then given $w \in T^a(U)$, π maps (a, w) to a and so π^* maps the cotangent space to U at a to the cotangent space at (a, w) to $T^{\cdot}(U)$:

$$\pi^*: T^a U \longrightarrow T^{(a,w)}(T^{\cdot} U).$$

Define the one-form ω on $T^{\cdot} U$ by $\omega(c) = (c, \pi^*(w))$ where $c = (a, w)$ is a point of $T^{\cdot} U$ (that is, $a \in U$ and $w \in T^a U$), and $\omega(c)$ is a point of $T^{\cdot}(T^{\cdot} U)$, since $c \in T^{\cdot} U$ and $\pi^* w \in T^c(T^{\cdot} U)$. Let us find what this invariant description becomes in terms of coordinates $\{q^i\}$ in U, and $\{q^i \circ \pi, p_i\}$ in $T^{\cdot} U$. We can always write w as $d_a f$ for some smooth function f; then

$$w = d_a f = \sum \left.\frac{\partial f}{\partial q^i}\right|_a d_a q^i = \sum p_i(c) d_a q^i .$$

Hence

$$\pi^* w = \sum p_i(c)\, d_{(a,w)}(q^i \circ \pi) = \sum p_i(c)\, d_c(q^i \circ \pi) ,$$

so

$$\omega(c) = (c, \sum p_i(c) d_c(q^i \circ \pi)).$$

Thus by abuse of notation $\omega = \sum p_i d(q^i \circ \pi) = \sum p_i dq^i$, which is the same form we have been working with all along. In particular, $\Omega = d\omega$, having been described invariantly, is independent of the particular coordinates we use. But our smooth bijective point transformation φ may be interpreted as nothing but a change of coordinates: if $\{q^i\}$ is a coordinate system on U, then so is $\{q^i \circ \varphi\}$. To say that Ω remains invariant under φ is to say that Ω is the same whether expressed in terms of $\{q^i \circ \varphi\}$ or $\{q^i\}$. Hence φ is a canonical transformation.

For those who like to get their hands dirty, here is a direct proof that φ is canonical: let $\{P_i, Q^i\}$ be the new coordinates on $T^* U$ induced by φ; then

$$dq^i = \sum_u \frac{\partial q^i}{\partial Q^j} dQ^j = \sum_j a^i_j \, dQ^j ,$$

$$p_i = \frac{\partial}{\partial q^i} = \sum_j \frac{\partial}{\partial Q^j} \frac{\partial Q^j}{\partial q^i} = \sum_j p_j b_i^j ,$$

where b_i^j and a_j^i are in fact inverse matrices. Then

$$\omega = \sum_i p_i \wedge dq^i = \sum_i \Big(\sum_j b_i^j P_j\Big) \wedge \Big(\sum_k a_k^i \, dQ^k\Big)$$

$$= \sum_{j,k} \Big(\underbrace{\sum_i b_i^j a_k^i}_{\delta_j^k}\Big) P_j \wedge dQ^k = \sum_j P_j \wedge dQ^k .$$

so φ is indeed canonical.

Definition. A smooth family of maps φ_t is called an infinitessimal canonical transformation if the induced coordinates $\{P_i(t), Q^i(t)\}$ in phase space satisfy

$$\frac{d}{dt}\left[\sum_{i=1}^{n} dP_i \wedge dQ^i\right]_{t=0} = 0.$$

Theorem. Every motion in phase space satisfying Hamilton's equation is an infinitessimal canonical transformation.

Proof. We will prove more; in fact we will show that the map which takes every point of phase space at $t = 0$ to the point representing the corresponding state of the system at time t is a canonical transformation. For this we calculate:

$$\frac{d}{dt}\Big(\sum_i dP_i \wedge dQ^i\Big) = \sum_i d\Big(\frac{dP_i}{dt}\Big) \wedge dQ^i + \sum_i dP_i \wedge d\Big(\frac{dQ^i}{dt}\Big)$$

(since if f is a function $\frac{\partial}{\partial t}(df) = d(\frac{\partial f}{\partial t})$ is easy to derive)

$$= \sum_i -d\Big(\frac{\partial H}{\partial Q^i}\Big) \wedge dQ^i + \sum_i dP_i \wedge d\Big(\frac{\partial H}{\partial P_i}\Big)$$

$$= \sum_i -\Big(\sum_j \frac{\partial^2 H}{\partial Q^i \partial Q^j} dQ^j \wedge dQ^i + \sum_j \frac{\partial^2 H}{\partial Q^i \partial P_j} dP_j \wedge dQ^i\Big)$$

$$+ \sum_i \Big(\sum_j \frac{\partial^2 H}{\partial P_i \partial Q^j} dP_i \wedge dQ^j + \sum_j \frac{\partial^2 H}{\partial P_i \partial P_j} dP^i \wedge dQ^j\Big).$$

Since mixed partial derivatives of smooth functions are equal, the second and third terms cancel. But the first and fourth terms are both zero, since $dQ^i \wedge dQ^j = -dQ^j \wedge dQ^i$. Hence $\frac{d}{dt}\big(\sum_i dP_i \wedge dQ_i\big) = \frac{d}{dt}\Omega$ is zero for all times t, so the motion of the system to time t is a canonical transformation for any t.

We sketch another proof of the preceding theorem. Let φ be a smooth map of an open set in $\mathbb{R}^2$ into M, where M is now viewed as any 2n-dimensional manifold. Let the coordinates on $\mathbb{R}^2$ be u and v, and the coordinates on M the usual $\{q^i, p_i\}$. It is easy to see that any 2-form ω on M is determined by the set $\{\varphi^* \omega\}$ for all possible such φ. But

$$\varphi^*(\Sigma\, dp_i \wedge dq^i) = \sum_i d(p_i\varphi) \wedge d(q^i\varphi) = \sum_i \left(\frac{\partial p_i\varphi}{\partial u} du + \frac{\partial p_i\varphi}{\partial v} dv\right) \wedge \left(\frac{\partial q^i\varphi}{\partial u} du + \frac{\partial (q^i\varphi)}{\partial v} dv\right)$$

$$= \sum_i \left(\frac{\partial(p_i\varphi)}{\partial u} \frac{\partial(q^i\varphi)}{\partial v} - \frac{\partial(q^i\varphi)}{\partial u} - \frac{\partial(p_i\varphi)}{\partial v}\right) du \wedge dv .$$

The coefficient of $du \wedge dv$ in this formula is called $[u,v]$, the <u>Lagrange bracket</u> of u and v, to prove the theorem, we must show that $d/dt[u,v] = 0$ at zero for all φ. We do the calculation only in the case where M is 2-dimensional, with coordinates p and q:

$$\frac{d}{dt}\left(\frac{\partial p}{\partial u} \frac{\partial q}{\partial v}\right) = \frac{\partial}{\partial u}\left(-\frac{\partial H}{\partial q}\right)\frac{\partial q}{\partial v} + \frac{\partial p}{\partial u} \frac{\partial}{\partial v}\left(\frac{\partial H}{\partial p}\right) \quad \text{(interchanging differential operators)}$$

$$= \frac{\partial^2 H}{\partial p \partial q} \frac{\partial p}{\partial u} \frac{\partial q}{\partial v} - \frac{\partial^2 H}{\partial q \partial q} \frac{\partial q}{\partial u} \frac{\partial q}{\partial v} + \frac{\partial^2 H}{\partial p \partial q} \frac{\partial p}{\partial u} \frac{\partial q}{\partial v} + \frac{\partial^2 H}{\partial p \partial p} \frac{\partial p}{\partial u} \frac{\partial p}{\partial v} .$$

The first and third terms above cancel; and when we subtract $d/dt\left(\frac{\partial p}{\partial v} \frac{\partial q}{\partial u}\right)$ corresponding terms also cancel each other, so the result is zero.

§20. Symplectic Spaces

We now turn to the problem of finding a standard way of writing 2-forms on M. It will turn out that, under suitable conditions, any closed and non-degenerate 2-form can be expressed as $\Sigma\, dp_i \wedge dq^i$ for some set of coordinates $\{p_i, q^i\}$. We look first at the situation on a vector space (i.e., on a single fiber of phase space).

Theorem. Let V be a finite-dimensional vector space, $\omega \in \Lambda_2(V^*)$. Then there is a basis $\{e_1, \ldots, e_m\}$ of V and an integer r such that

$$\omega = e^1 \wedge e^{n+1} + \ldots + e^r \wedge e^{2r} .$$

Proof. Regard ω as an alternating bilinear form on V. We may assume $\omega \neq 0$; then we can find linearly independent vectors e_1, e_2 with $\omega(e_1, e_2) \neq 0$. By scalar multiplication we can adjust e_1 and e_2 so that $\omega(e_1, e_2) = 1$. Now let S be the subspace of all $v \in V$ satisfying $\omega(e_1, v) = \omega(e_2, v) = 0$. Calculation shows that no linear combination of e_1 and e_2 lies in S. Furthermore, V is spanned by S, e_1, and e_2. For let z be any vector of V; we wish to find numbers x and y such that $v = z - xe_1 - ye_2$ lies in S. To force $\omega(e_1, v) = 0$ we must have $\omega(e_1, z - xe_1 - ye_2) = \omega(e_1, z) - y = 0$. Thus $y = \omega(e_1, z)$; similarly, we can take $x = -\omega(e_2, z)$. This accomplished, we now apply the same technique to the form ω restricted to S. We find $e_3, e_4 \in S$ and a subspace $S' \subsetneq S$ such that no linear combination of e_3 and e_4 lies in S', but e_3, e_4 and S' span S. Continuing in this fashion, we eventually find $S^{(k)}$ on which ω is identically zero. Then, choosing any basis $\{e_{2k+3}, \ldots, e_m\}$

for $S^{(k)}$, we get a basis $\{e_1, \ldots, e_m\}$ for V with the property that $\omega(e_1, e_2) = \omega(e_3, e_4) = \ldots = \omega(e_{2k+1}, e_{2k+2}) = 1$, and all other $\omega(e_i, e_j) = 0$ (except for reversals of the above, e.g., $\omega(e_2, e_1) = -1$). This means that $\omega = e^1 \wedge e^2 + e^3 \wedge e^4 + \ldots + e^{2k+1} \wedge e^{2k+2}$. Renumbering the e's gives us the desired formula, as in the theorem.

Each ω determines a linear map $\omega^\flat : V \longrightarrow V^*$ given by $[\omega^\flat(v)]v' = \omega(v, v')$. We say ω is <u>non-degenerate</u> if $\omega^\flat$ is an isomorphism. Since V and V^* are finite-dimensional, this is equivalent to saying that $\omega^\flat$ has zero null-space; in other words, $\omega(v, v') = 0$ for all v' implies $v = 0$. Using now the canonical form given in the theorem, we derive

<u>Corollary 1</u>. If ω is non-degenerate and in the form given by the theorem, then V is $2r$-dimensinal.

For if $m > 2r$, $\omega(e_{2r+1}, v) = 0$ for all $v \in V$.

More computation with the canonical form establishes

<u>Corollary 2.</u> If V is $2n$-dimensional, ω is non-degenerate if and only if $\omega \wedge \omega \wedge \ldots \wedge \omega$ (n times) $= 0$.

<u>Corollary 3.</u> The integer r in the theorem is determined by

$$\omega^r \neq 0 \ , \quad \omega^{r+1} = 0 \, .$$

The number $2r$ is called the <u>rank</u> of ω.

<u>Definition.</u> A <u>symplectic vector space</u> is a finite-dimensional vector space V with a non-degenerate form $\omega \in \Lambda_2(V^*)$.

We have proved that every symplectic vector space has dimension 2n for some integer n and possesses a <u>symplectic basis</u> $\{e_i\}$ for which $\omega = \sum_{i=1}^{n} e^i \wedge e^{n+i}$. In this basis, the matrix of the bilinear form ω is

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Furthermore,

$$\begin{aligned}(\omega^\flat e_i)(e_j) &= \omega(e_i, e_j) \\ &= \sum_{1 \le k \le n} \tfrac{1}{2}(e^k \otimes e^{n+k} - e^{n+k} \otimes e^k)(e_i, e_j) .\end{aligned}$$

Hence $\omega^\flat(e_i) = \frac{1}{2} e^{n+i}$, $1 \le i \le n$. Similarly, $\omega^\flat(e_{n+i}) = -\frac{1}{2} e^i$. If ω is non-degenerate, $\omega^\flat$ is an isomorphism, and so is $2\omega^\flat$.

This map $2\omega^\flat : V \longrightarrow V^*$ is given in this basis by the convenient formulas

$$(2\omega^\flat)(e_i) = e^{n+i} ,$$

$$(2\omega^\flat)(e_{n+i}) = -e^i , \quad i = 1, \ldots, n.$$

We let $\omega^\# = (2\omega^\flat)^{-1}$ be the inverse map.

§21 Hamilton's equations

Now apply this machinery to $T_\cdot(M)$. Here $M = T^\cdot U$ is our 2n-dimensional phase space, and we are given a non-degenerate closed 2-form ω on M. This form ω yields maps $2\omega^\flat$ and $\omega^\#$ on each tangent space of M, and hence gives a bundle map $2\omega^\flat : T_\cdot(M) \longrightarrow T^\cdot(M)$. Also, the Hamiltonian H is a function on M, and thus gives rise to a one-form dH on M. Here is the diagram:

$$\begin{tikzcd}
T_{\cdot}(M) \arrow[rr, shift left, leftarrow, "\omega^{\#}"] \arrow[rr, shift right, "2\omega^{\flat}"'] \arrow[dr] & & T^{\cdot}(M) \arrow[dl, shift right] \\
& M \arrow[ur, shift right, "dH"'] &
\end{tikzcd}$$

Explicitly, if c is a point of M, we have the map $2\omega^{\flat}: T_c M \to T^c M$, where we have identified V above with $T_c M$, V^* with $T^c M$. This allows us to pass from vector fields to one-forms and vice versa, since $\omega^{\flat}$ is an isomorphism. Specifically, $\omega^{\#}(dH)$ is a vector field on M. Identifying the $\{dq^i\}$ with $e^1, \ldots, e^n$ and the $\{dp_i\}$ with $e^{n+1}, \ldots, e^n$, we compute

$$dH = \sum \frac{\partial H}{\partial q^i} dq^i + \sum \frac{\partial H}{\partial p_i} dp_i \;;$$

that is,

$$dH = \left(\frac{\partial H}{\partial q^i}, \ldots, \frac{\partial H}{\partial q^n}, \frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n} \right).$$

Therefore

$$\begin{aligned}
\omega^{\#}(dH) &= \left(\frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q^1}, \ldots, -\frac{\partial H}{\partial q^n} \right) \\
&= \sum \frac{\partial H}{\partial q^i} \frac{\partial}{\partial q^i} - \sum \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} .
\end{aligned}$$

The vector field $X = \omega^{\#}(dH)$ determines a system of differential equations on M, namely $dc/dt = X(c)$, where c is a path in M. This equation means that the curve c threads its way through the tangent vector field X on M in such a way that the tangent vector to c at any point is exactly the same as the value of the field X at that point. But in coordinates, the paths which satisfy the equation $dc/dt = X(c)$ are those which satisfy Hamilton's equations:

$$\frac{\partial q^i}{\partial t} = \frac{\partial H}{\partial p_i}\, , \quad \frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial q^i}$$

if $X = \omega^{\#}(dH) = (\frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q^i}, \ldots, -\frac{\partial H}{\partial q^n})$. In other words, we have reached an invariant way of stating Hamilton's equations on a local manifold.

§22 Symplectic manifolds

Generalizing the results of our last lecture, we define a symplectic manifold to be a manifold possessing a closed non-degenerate two-form ω. We have seen that $\omega \in \Lambda_2(V^*)$ may be viewed as an alternating bilinear function on $V \times V$ to $\mathbb{R}$; thus if v and v' are vectors of V, $\omega(v, v')$ is a real number. We will say that a linear transformation $f: V \longrightarrow V$ is symplectic if it is one-to-one onto and preserves the form ω; that is, $f^*\omega = \omega$, or $\omega(fv, fv') = \omega(v, v')$ for all $v, v' \in V$. The set of all such f is called the symplectic group; it is in fact a group in the mathematical sense, since the operation of composition is associative, has inverses, and has a unit element. In particular though, a symplectic transformation is a linear transformation on a finite-dimensional vector space, and thus may have eigenvalues.

Theorem. If λ is an eigenvalue of a symplectic transformation, then so are $\bar{\lambda}$ (the complex conjugate of λ), $1/\lambda$ and $1/\bar{\lambda}$; in particular $\lambda \neq 0$.

Proof. Over the complex numbers, eigenvalues always exist; so our first task is to create a complex vector space out of the real vector space V; that is, to find a way of multiplying by complex as well as real scalars. It is not hard to see that taking the tensor product $\mathbb{C} \otimes V = \tilde{V}$ results in a complex vector space $\tilde{V}$, where the product of a vector $\sum c_i \otimes v_i$ by a complex scalar c is $c(\sum c_i \otimes v_i) = \sum cc_i \otimes v_i$. Also, the real dimension of $\tilde{V}$ is twice that of V: if $\{e_1, \ldots, e_m\}$ is a basis of V, then since $\{1, i\}$ is a basis of $\mathbb{C}$ considered as a real vector space, a basis of $\tilde{V}$ as a real vector space is $1 \otimes e_1, \ldots, 1 \otimes e_m, i \otimes e_1, \ldots, i \otimes e_m$; there are $2m$ of these vectors. But the m vectors $1 \otimes e_1, \ldots, 1 \otimes e_m$ form a basis of $\tilde{V}$ over $\mathbb{C}$ since scalar multiplication by i converts $1 \otimes e_j$ into $i \otimes e_j$. We now extend ω to a form $\tilde{\omega}$ on $V \otimes \mathbb{C}$, defining

$$\tilde{\omega}(c \otimes v, c' \otimes v') :: cc'\omega(v, v')$$

for any two complex numbers c and c' and any two vectors $v, v' \in V$, and extending this definition by linearity. Similarly, each linear $f: V \longrightarrow V$ goes over to $\tilde{f} = 1 \otimes f: \mathbb{C} \otimes V \longrightarrow \mathbb{C} \otimes V$. Moreover, the formula $\tilde{\omega}(\tilde{f}v, \tilde{f}v') = \tilde{\omega}(v, v')$ still holds; that is, if f is symplectic, $\tilde{f}$ is symplectic with respect to $\tilde{\omega}$.

Now suppose u is an eigenvector corresponding to λ : $u \neq 0$, $f(u) = \lambda u$. If λ were zero, u would be in the null-space of f, contradicting the assumption that f was one-to-one. But then if u' is any other vector, $\tilde{\omega}(u, u') = \tilde{\omega}(\tilde{f}u, \tilde{f}u') = \tilde{\omega}(\lambda u, \tilde{f}u') = \tilde{\omega}(u, \lambda \tilde{f}u')$, so $\tilde{\omega}(u, u' - \lambda \tilde{f}(u')) = 0$. Now if the map taking $u' \rightsquigarrow u' - \lambda \tilde{f}(u')$ were onto, every vector w of $\tilde{V}$ could be written in the form $w = v - \lambda \tilde{f}(v)$. Then we'd have $\omega(u, w) = 0$ for every $w \in \tilde{V}$, which would mean that $u = 0$ since ω is non-singular. This is a contradiction. Hence the map mentioned above is not onto; since it's a linear transformation between vector spaces of the same dimension, it's also not one-to-one. Thus there is a u' with $u' - \lambda \tilde{f}(u') = 0$; this says that $\tilde{f}(u') = \frac{u'}{\lambda}$, so $\frac{1}{\lambda}$ is an eigenvalue of $\tilde{f}$, and hence an eigenvalue of f. Also $\tilde{f}(\bar{v}) = \overline{\tilde{f}(v)}$ by the definition of f, so $\tilde{f}(u) = \lambda u$ implies $\overline{\tilde{f}(u)} = \overline{\lambda u}$ which implies $\tilde{f}(\bar{u}) = \bar{\lambda}\bar{u}$. Hence $\bar{\lambda}$ is an eigenvalue, so by what we've already proved, $\frac{1}{\bar{\lambda}}$ is an eigenvalue. This completes the proof of the theorem.

§23 <u>The Poincaré Lemma.</u>

We return to the problem of expressing our closed form ω in some system of coordinates as $\omega = \sum dp_i \wedge dq^i$ in some <u>region</u>, not just at a point. We first prove the Poincaré lemma:

<u>Theorem</u> Let U be an open ball in $\mathbb{R}^n$. Let ω be a closed k-form on U; that is, $d\omega = 0$. Then there is a $(k-1)$-form η on U such that $d\eta = \omega$. (succinctly: closed forms are exact on U).

<u>Proof.</u> We will first derive a new formula for the differential d, which makes $(k+1)$-forms out of k-forms. We will then use the assumption on U to define a new map s which makes $(k-1)$-forms out of k-forms

for each k, such that $ds(\omega) + sd(\omega) = \omega$ for every form ω. If then we have an ω with $d\omega = 0$, it will follow that $\omega = d(s\omega)$, showing that ω is exact. We will also let V denote the tangent space (at any point) of U

A k-form ω may be regarded as a smooth map from U to $\Lambda_k(V^*)$, the space of alternating k-tensors on V. Thus for each $u \in U$, ω_u is an alternating k-tensor: $v_1, \ldots, v_k \in V$ implies that $\omega_u(v_1, \ldots, v_k) \in \mathbb{R}$. Write $\omega(u, v_1, \ldots, v_k) = \omega_u(v_1, \ldots, v_k)$; then ω is a function smooth in the first argument, and linear and alternating in the last k arguments.

Suppose f is a smooth real-valued function on U. We define a new function $Df: U \times V \longrightarrow \mathbb{R}$ by letting $Df(u,v) = \langle d_u f, v \rangle$; that is, $Df(u,v) = \left.\frac{d(f \circ \tilde{v})}{dt}\right|_{t=0}$ where v is the path defined by $\tilde{v}(t) = u+tv$. Hence Df is nothing more than the directional derivative of f in the direction v at the point u. Now if f happens to be a function of other variables as well, we can still form Df by ignoring those other variables as we take the derivative, and then putting them back: thus if $f = f(u, w_1, \ldots, w_r)$,

$$Df(u, v, w_1, \ldots, w_r) = (d/dt) f(u+tv, w_1, \ldots, w_r)\Big|_{t=0}$$

Notice that Df is a linear function of v; if also f happens to be a linear function (in u), $Df(u,v) = f(v)$.