

Definition. An exterior  $k$ -form  $\omega$  on  $U$  ( $k = 0, 1, \dots, n$ ) is a smooth cross-section of  $\Lambda^k U$

$$\begin{array}{c} \Lambda^k U \\ \downarrow \pi \\ U \end{array}$$

Thus a  $k$ -exterior form  $\omega$  on  $U$  is a smooth map  $U \xrightarrow{\omega} \Lambda^k U$  of the form  $a \rightsquigarrow (a, \tilde{\omega}a)$  where  $\tilde{\omega}a \in \Lambda_k(T^a U)$ .

Note that a 1-form by this definition coincides with our previous definition of a 1-form, and that a 0-form is just a smooth map from  $U$  to  $\mathbb{R}$ .

Definition.  $\Omega^k(U) =$  the set of all  $k$ -forms  $\omega$  on  $U$ .

We remark that

1)  $\Omega^k(U)$  is a vector space over  $\mathbb{R}$ , with vector space operations as follows: if  $\omega \in \Omega^k(U)$ ,  $\alpha \in \mathbb{R}$  and  $\omega: a \rightsquigarrow (a, \tilde{\omega}a)$ , then

$$\alpha\omega: a \rightsquigarrow (a, \alpha(\tilde{\omega}a)) ; \omega_1 + \omega_2: a \rightsquigarrow (a, \tilde{\omega}_1 a + \tilde{\omega}_2 a).$$

2)  $\Omega^k(U)$  is an  $\mathcal{F}$ -module, where  $\mathcal{F}$  = the ring of smooth functions  $f: U \rightarrow \mathbb{R}$ . The module action is given by  $f\omega: a \rightsquigarrow (a, f(a)\tilde{\omega}a)$ .

3) For each  $k$  and  $m$ , there is an exterior product

$$\Omega^k(U) \times \Omega^m(U) \longrightarrow \Omega^{k+m}(U)$$

$$(\omega, \eta) \rightsquigarrow \omega \wedge \eta$$

defined by

$$\omega \wedge \eta: a \rightsquigarrow (a, \tilde{\omega}(a) \wedge \tilde{\eta}(a)).$$

This is well-defined since  $\tilde{\omega}(a)$  and  $\tilde{\eta}(a)$  are both elements of  $\Lambda(T^a U)$ .

The exterior product is associative and bilinear. Thus the  $\Omega^k(U)$  form a graded algebra  $\Omega^*(U)$  over  $\mathcal{F}$ .

Since  $\Lambda^k(U)$  has as basis all  $k$ -fold wedge products of the  $dq^i$ , where  $q^1, \dots, q^n$  are coordinates in  $U$ , any  $k$ -form  $\omega$  is of the form

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} f^{i_1 \dots i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k},$$

where  $f^{i_1 \dots i_k} : U \rightarrow \mathbb{R}$ . If the  $p^{i_1 \dots i_k}$  are the coordinates corresponding to the  $dq^{i_1} \wedge \dots \wedge dq^{i_k}$ , then

$$f^{i_1 \dots i_k} = p^{i_1 \dots i_k} \cdot \omega.$$

Given a smooth map  $U \xrightarrow{\varphi} U'$ , we have an induced map  $T^a U \xleftarrow{\varphi^*} T^a U'$ , which in turn determines, via the properties of the exterior algebra, a map  $\Omega^k U \xleftarrow{\varphi^*} \Omega^k U'$  such that  $\varphi^*$  is linear and

$$\varphi^*(\omega \wedge \eta) = (\varphi^* \omega) \wedge (\varphi^* \eta).$$

Then if  $q^1, \dots, q^n$  are coordinates in  $U$  and  $r^1, \dots, r^n$  in  $U'$ , it will follow that

$$\varphi^*(dq^1 \wedge \dots \wedge dq^n) = \det\left(\frac{\partial q^i}{\partial r^j}\right) dr^1 \wedge \dots \wedge dr^n,$$

which is just the usual change-of-coordinates rule.

Now we define the basic operations of exterior differentiation of forms.

Theorem. There exists a unique linear map

$$d: \Omega^k(U) \longrightarrow \Omega^{k+1}(U) \quad \text{for each } k$$

such that

- 1) for  $f \in \Omega^0$ ,  $df =$  usual differential of  $f$
- 2)  $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$ , where  $k =$  degree of  $\omega$  [i. e.,

every time  $d$  is moved past something of degree 1, the sign changes]

$$3) \quad dd\omega = 0 \text{ for all } \omega$$

( $d$  is called the exterior derivative)

Proof. If  $d$  exists, then by linearity and (1)-(3),

$$\begin{aligned} d\left(\sum f^{i_1 \dots i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k}\right) \\ &= \sum d(f^{i_1 \dots i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k}) \\ &= \sum df^{i_1 \dots i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k}, \end{aligned}$$

since all other summands have a factor of the form  $ddq^i$  and hence are zero by (3).

Since every  $k$ -form  $\omega$  can be written as

$$\omega = \sum f^{i_1 \dots i_k} dq^{i_1} \dots dq^{i_k}$$

for smooth functions  $f^{i_1 \dots i_k}$ , the above shows that  $d$ , if it exists, must be unique, since it is determined by its effect on 0-forms, which is specified by (1). But the above expression also defines a function  $d: \Omega^k(U) \longrightarrow \Omega^{k+1}(U)$  for each  $k$ . It is easily verified that this  $d$  is linear and satisfies (1)-(3).

