# GEOMETRICAL MECHANICS

## Part II

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Introduction to Part II

This sequel to Part I completes the notes of my two-quarter course on Geometrical Mechanics, except for the final section of the course which discussed Relativity Theory, the Schwarzschild metric, and the relativistic explanation of the advance in the perihelion of Mercury. (These lectures have not been reduced to written form.)

These notes have many of the imperfections of a first course on a new subject. Here the new subject is the use of modern geometrical ideas in the long-stagnant treatment of classical mechanics. The initiative of George Mackey has been vital for this subject, and the books by Ralph Abraham and Schlomo Sternberg are excellent guides. A few of the topics covered here are apparently not to be found in this form in the literature: The treatment of the Legendre transformation (§9 of Chapter I), the conceptual treatment of the generating functions for canonical transformations (§26 of Chapter III and §44 of Chapter VI), the description of manifolds by means of germs (Chapter IV, §30) and the geometric description of the characteristics of first order partial differential equations (Chapter VI, §46). This, with the material on contact transformations, may suggest how much of classical Mathematics stands in need of modern conceptual formulation.

I am much indebted to the students whose notes have improved and codified my lectures, and to René Thom for permission to include the material of his guest lectures.

The University of Chicago

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# CHAPTER IV. MANIFOLDS

## 29 Topological Spaces

To define manifolds, we first review the basic properties of topological spaces.

Definition. A topological space is a pair $(X, t)$ where $X$ is a set and $t$ is a collection of subsets of $X$ such that:

$1^\circ$ $\emptyset \in t$, $X \in t$;

$2^\circ$ $U \cap V \in t$ whenever $U \in t$ and $V \in t$;

$3^\circ$ If $\{U_\alpha\}_{\alpha \in A}$ is a collection of subsets of $X$ such that $U_\alpha \in t$ for each $\alpha$, then $\bigcup_{\alpha \in A} U_\alpha \in t$.

Here $t$ is called the topology of $X$. The sets in $t$ are called open sets. A subset $F$ of $X$ is called closed if $X - F \in t$, where $X - F = \{x \in X \mid x \notin F\}$ is the complement of $F$ in $X$. We will often use just $X$ to refer to the topological space $(X, t)$ when it is clear what topology on $X$ is intended.

Example. $\mathbb{R}^n$ together with the subsets which we have previously called open is a topological space.

If $(X, t)$ and $(X', t')$ are topological spaces, a function $f: X \longrightarrow X'$ is continuous if $f^{-1}(V) \in t$ whenever $V \in t'$, where the set $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is the inverse image of $V$ under $f$.

The function $f: X \longrightarrow X'$ is a homeomorphism if it is a bijection (one-to-one onto) and both $f$ and $f^{-1}$ are continuous.

A neighborhood of a point $x \in X$ is any open set in $X$ containing $x$.

A function $f: X \longrightarrow X'$ is continuous at $x$, where $x \in X$, if for every neighborhood $V$ of $f(x)$ in $X'$, there exists a neighborhood $U$ of $x$ in $X$ with

$$f(U) \subset V \qquad (\text{i.e.}, \; U \subset f^{-1}(V))$$

It is easy to show that $f: X \longrightarrow X'$ is continuous if and only if $f$ is continuous at every point $x \in X$.

Examples:

$1^{\circ}$ If $(X, t)$ is a topological space and $S$ is any subset of $X$, let

$$t' = \{U \cap \rho \mid U \in t\}.$$

Then $t'$ is a topology for $S$, called the relative topology.

$2^{\circ}$ If $(x, t)$ is a topological space and the function $X \xrightarrow{p} S$ maps $X$ onto the set $S$, let

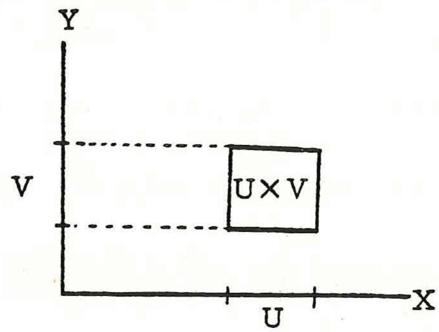
$$t' = \{V \subset S \mid p^{-1}(V) \in t\}.$$

Then $t'$ is a topology, called the quotient topology, for $S$, and $(X, t) \xrightarrow{p} (S, t')$ is a continuous map.

$3^{\circ}$ If $(X, t)$ and $(Y, t')$ are topological spaces, let $X \times Y$ denote the ordinary cartesian product of the sets $X$ and $Y$. Let

$\bar{t}_o = \{U \times V \mid U \in t, \; V \in t'\}$,

$\bar{t} =$ all subsets of $X \times Y$ which are unions of sets in $\bar{t}_o$.




Then $(X \times Y, \bar{t})$ is a topological space, and $\bar{t}$ is called the product topology.

We have the two projection mappings onto $X$ and $Y$:

$$p: X \times Y \longrightarrow X \qquad q: X \times Y \longrightarrow Y$$
$$(x, y) \longmapsto x \qquad (x, y) \longmapsto y$$

These are both continuous when $X \times Y$ is given the product topology.

Definition: If $(X, t)$ is a topological space, a basis for $(X, t)$ (or a "basis for the open sets of $(X, t)$") is a collection $t_o$ of open subsets of $X$ such that every member of $t$ is a union of members of $t_o$.

Thus in Example $3^o$, $\bar{t}_o$ is a basis for $(X \times Y, \bar{t})$.

A topological space is separable if it has a countable basis.

A sub-basis $t_1$ for $(X, t)$ is a collection of subsets of $X$ such that the set $t_o$ of all finite intersections of members of $t_1$ forms a basis for $X$. Given any collection $t_1$ of subsets of a set $X$ such that $X = \bigcup_{t_1} U$, there exists a unique topology $t$ for $X$ having $t_1$ as a sub-basis -- namely, $t$ consists of all unions of finite intersections of members of $t_1$.

If $\{(X_\alpha, t_\alpha)\}_{\alpha \in A}$ is a family of topological spaces, let $X = \prod_\alpha X_\alpha$ be the cartesian product of the sets $X_\alpha$ and let $P_\alpha : X \longrightarrow X_\alpha$ be the projection onto the $\alpha^{th}$ coordinate space. Let

$$t_1 = \bigcup_{\alpha \in A} \{P_\alpha^{-1}(V) \mid V \in t_\alpha\} .$$

The topology $t$ for $X$ having $t_1$ as sub-basis is called the <u>product topology.</u> Then each $P_\alpha : (X, t) \longrightarrow (X_\alpha, t_\alpha)$ is continuous. If $(X', t')$ is another space, and if for each $\alpha \in A$ we are given a continuous map $f_\alpha : (X', t') \longrightarrow (X_\alpha, f_\alpha)$, there exists a unique function $f : X' \longrightarrow X$ such that $P_\alpha f = f_\alpha$ for each $\alpha$, since the set $X$ is the set product of the $X_\alpha$. Then if $V \in t_1$ -- say $V = P_\alpha^{-1}(U_\alpha)$ where $U_\alpha$ is open in $X_\alpha$, it follows that $f^{-1}V = f^{-1}P_\alpha^{-1}U_\alpha = (P_\alpha f)^{-1}U_\alpha = f_\alpha^{-1}U_\alpha$ is open in $X'$. Since $t_1$ is a sub-basis for $(X, t)$, it follows that $f$ is continuous (This means that $(X, t)$ is the product of the $(X_\alpha, t_\alpha)$ in the category of all topological spaces.)

Suppose the $X_\alpha$ as above are disjoint (if not, take disjoint homeomorphic copies). Then we can topologize their <u>disjoint union</u> $Y = \coprod_\alpha X_\alpha$ as follows:

$$U \subset Y \text{ is open in } Y \Longleftrightarrow U \cap X_\alpha \text{ is open in } X_\alpha \text{ for each } \alpha.$$

In a fashion similar to that above, if $(X', t')$ is another topological space and $g_\alpha : X_\alpha \longrightarrow X'$ is a continuous map for each $\alpha$, then there exists a unique continuous map $g : X \longrightarrow X'$ such that $g_\alpha = gq_\alpha$, where $q_\alpha : X_\alpha \longrightarrow X$ is the injection of $X_\alpha$ into the disjoint union. This means that $Y$ is the "coproduct" of the $(X_\alpha, t_\alpha)$ in the category of all topological spaces.

Suppose we are given a set $X$ and subsets $X_\alpha$ each with a topology $t_\alpha$ such that

1° $X = \bigcup_\alpha X_\alpha$.

2° $Y_{\alpha\beta} = X_\alpha \cap X_\beta$ is open in both $X_\alpha$ and $X_\beta$, and the relative topologies on $Y_{\alpha\beta}$ induced from $(X_\alpha, t_\alpha)$ and $(X_\beta, t_\beta)$ coincide.

Then $X$ has a topology

$$t = \{U \subset X \mid U \cap X_\alpha \in t_\alpha \text{ for all } \alpha\}.$$

This situation can be expressed by the statement that $X$ is the coequalizer, in the category of topological spaces and continuous maps, of the maps

$$\coprod_{\alpha,\beta} Y_{\alpha\beta} \rightrightarrows \coprod_\alpha X_\alpha,$$

where one map injects $Y_{\alpha\beta}$ into $X_\alpha$, the other into $X_\beta$.

An <u>open covering</u> for a topological space $(X,t)$ is a collection of open sets of $X$ whose union is $X$. If $\{U_\alpha\}$ is an open covering for $(X,t)$, it is easily checked that a function $f: (X,t) \longrightarrow (Y,t')$ is continuous if and only if $f|U_\alpha: U_\alpha \longrightarrow Y$ is continuous in the relative topology of $U_\alpha$ for each $\alpha$.

A topological space $X$ is Hausdorff if for every pair of points $x, y \in X$ there exist open sets $U, V \subset X$ with $x \in U$, $y \in V$ such that $U \cap V = \emptyset$.

30. Manifolds.

Let $X$ be a topological space, $x \in X$. Every function $f: U \longrightarrow \mathbb{R}$ such that $x \in U$ and $U$ is an open subset of $X$ determines the germ $f_x$ of $f$ at $x$, where $f_x = g_x$ if $g: V \longrightarrow \mathbb{R}$ and there exists $W \subset U \cap V$ such that $x \in W$, $W$ is open in $X$ and $f|W = g|W$.

Let $C_x$ denote the set of germs of all continuous functions to $\mathbb{R}$ defined on some neighborhood of $x$. $C_x$ is an algebra.

Definition. A loaded space is a triple $(X, t, G)$ where $(X, t)$ is a topological space and $G$ assigns to each point $x \in X$ a set $G_x$ of germs at $x$ (germs of the "good" functions).

Unless otherwise specified, we will assume that $G_x \subset C_x$. Often we will require that $G_x$ be an algebra.

Examples:

$1^{\circ}$ $X = U_o$ open in $\mathbb{R}^n$ (e.g., $X = \mathbb{R}$) and $G$ = germs of all $C^{\infty}$ functions at $x$. Call this loaded space $(U_o, C^{\infty})$.

$2^{\circ}$ If $(X, G)$ is a loaded space and $V$ is open in $X$, then $(V, G|V)$ is a loaded space.

$3^{\circ}$ Let $(X, t)$ be a topological space and $\mathcal{F}$ any set of continuous functions $f: X \longrightarrow \mathbb{R}$. Set $\mathcal{G}_x = \{f_x \mid f \in \mathcal{F}\}$. Then $(X, t, \mathcal{G})$ is a loaded space.

If $(X, \mathcal{G})$ is a loaded space and $U$ is open in $X$, define

$$\mathcal{G}(U) = \{f \mid f: U \longrightarrow \mathbb{R} \text{ continuous and } f_x \in \mathcal{G}_x, \text{ for all } x \in U\}.$$

$\mathcal{G}(U)$ has the <u>sheaf property</u>: if $U = \bigcup_{\alpha} V_{\alpha}$ where the $V_{\alpha}$ are open, then $f \in \mathcal{G}(U)$ if and only if for each $\alpha$, $f|V_{\alpha} \in \mathcal{G}(V_{\alpha})$.

If $(X, \mathcal{G})$ and $(Y, \mathcal{H})$ are loaded spaces, a <u>loaded map</u> $(X, \mathcal{G}) \xrightarrow{\varphi} (Y, \mathcal{H})$ is a map $X \xrightarrow{\varphi} Y$ such that

$1^{\circ}$ $\varphi$ is continuous,

$2^{\circ}$ $x \in X$, $h_{\varphi(x)} \in \mathcal{H}_{\varphi(x)}$ implies $(h\varphi)_x \in \mathcal{G}_x$.

Notice the similarity of this definition to that of a continuous map.

The following facts follow easily from the last definition:

$1^{\circ}$ The composite of loaded maps is loaded.

$2^{\circ}$ $V$ open in $Y$ implies $\varphi^{*}\mathcal{H}(V) \subset \mathcal{G}(\varphi V)$.

$3^{\circ}$ $\varphi$ is loaded if and only if at each $x \in X$, $\varphi$ is continuous and carries "good" germs at $\varphi(x)$ to "good" germs at $x$.

A <u>loaded isomorphism</u> is a loaded map $(X, \mathcal{G}) \xrightarrow{\varphi} (Y, \mathcal{H})$ such that

$1^{\circ}$ $X \xrightarrow{\varphi} Y$ is a topological isomorphism (i.e., a <u>homeomorphism</u>)

$2^{\circ}$ for each $x \in X$, the correspondence $\mathcal{H}_{\varphi(x)} \longrightarrow \mathcal{G}_x$ induced by $\varphi$ is one-to-one and onto.

<u>Definition.</u> A $C^\infty$ n-chart on $(X, \mathcal{G})$ consists of

$1^\circ$ an open set $U$ of $X$, called the <u>domain</u> of the chart,

$2^\circ$ a loaded isomorphism $(U, \mathcal{G}|U) \simeq (U_o, C^\infty)$, where $U_o$ is open in $\mathbb{R}^n$.

A $C^\infty$ <u>n-manifold</u> is a loaded space $(M, \mathcal{G})$ such that $M$ is Hausdorff and the domains of all $C^\infty$ n-charts on $(M, \mathcal{G})$ cover $M$. We will usually also require that $M$ be separable.

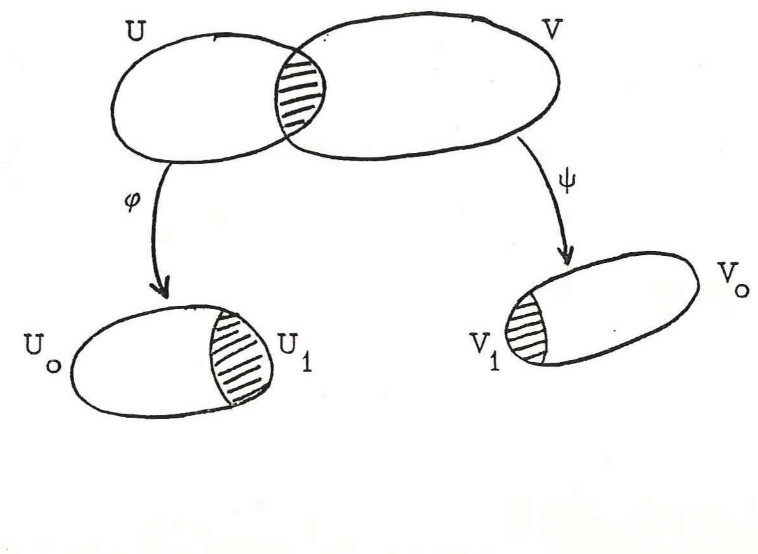
<u>Example:</u> Any open set in $\mathbb{R}^n$ is a $C^\infty$ n-manifold.

An <u>atlas</u> of a $C^\infty$ n-manifold $(M, \mathcal{G})$ is a set of n-charts whose domains cover $M$. The same manifold can have many atlases; the only "invariant" one is the maximal atlas (<u>all</u> charts).

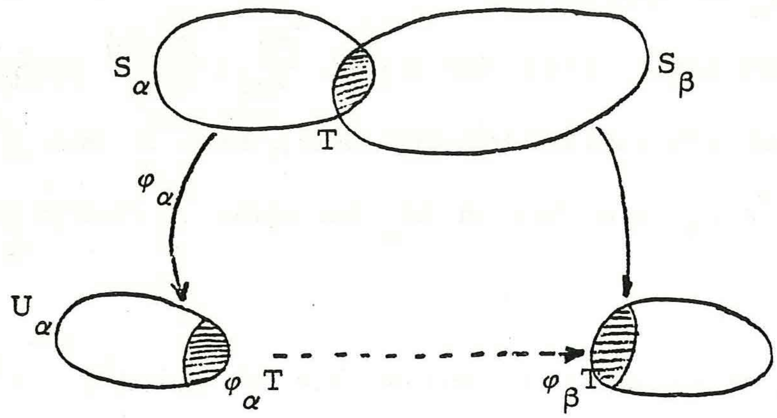
Uf $U \overset{\varphi}{\simeq} U_o$ and $V \overset{\psi}{\simeq} V_o$ are charts of the $C^\infty$ manifold $(M, \mathcal{G})$, then the induced map

$$U_1 \underset{\varphi}{\simeq} U \cap V \underset{\psi^{-1}}{\simeq} V_1$$

is $C^\infty$, for if $x^1, \dots, x^n$ are smooth coordinates on $V_1$, then it follows that each $x^i \circ \theta$ is smooth.



Suppose S is a set. A <u>chart</u> on S is a one-to-one function $S_\alpha \xrightarrow{\varphi_\alpha} U_\alpha$ where $S_\alpha \subset S$, $U_\alpha$ is open in $\mathbb{R}^n$, and $\varphi_\alpha$ maps $S_\alpha$ onto $U_\alpha$. Two charts $\varphi_\alpha$ and $\varphi_\beta$ are <u>compatible</u> if, for $T = S_\alpha \cap S_\beta$, $\varphi_\alpha T$ is open in $U_\alpha$, $\varphi_\beta T$ is open in $U_\beta$, and $(\varphi_\beta | T) \circ (\varphi_\alpha^{-1} | \varphi_\alpha T)$ is a $C^\infty$ map from $\varphi_\alpha T$ onto $\varphi_\beta T$.




If $\{(S_\alpha, \varphi_\alpha)\}$ is a family of pairwise compatible charts on S such that the $S_\alpha$ cover S, then the $\varphi_\alpha$ collectively determine a topology on S. If this topology is Hausdorff, then S becomes a manifold. More generally, we have the following theorem which is often used to construct a manifold from overlapping pieces $M_\alpha$ (especially with each $M_\alpha$ an open set in $\mathbb{R}^n$):

<u>Theorem.</u> If X is a set and $X = \bigcup_\alpha M_\alpha$ where each $M_\alpha = (M_\alpha, t_\alpha, \mathcal{G}^{(\alpha)})$ is a $C^\infty$ n-manifold such that for each $\alpha$ and $\beta$

1° $T_{\alpha\beta} = M_\alpha \cap M_\beta$ is open in both $M_\alpha$ and $M_\beta$;

2° if $x \in T_{\alpha\beta}$ and g is a real-valued function defined near x, then

$$g_x \in \mathcal{G}^{(\alpha)} \Longleftrightarrow g_x \in \mathcal{G}^{(\beta)} ;$$

$3^{o}$ if $U \subset T_{\alpha\beta}$, then $U \in t_\alpha \Longleftrightarrow U \in t_\beta$;

then

a) X has a topology, namely W is open in X if and only if for every $\alpha$, $W \cap M_\alpha$ is open in $M_\alpha$. (We've seen this part before. In particular, each $M_\alpha$ is open in X.)

b) X is a loaded space, where for $x \in M$, $\mathcal{G}_x = \mathcal{G}_x^{(\alpha)}$ with $x \in M_\alpha$.

c) If X is Hausdorff with the topology in (a), then X is a $C^\infty$ n-manifold. If $U \xrightarrow{\varphi} U_o$ is a chart in $M_\alpha$ for some $\alpha$, then it is also a chart in X.

If $U \xrightarrow{\varphi} U_o$ is a chart on the n-manifold M and $x^1, \ldots, x^n$ are coordinates on $U_o \subset \mathbb{R}^n$, then $q^i = x^i \circ \varphi$ are called <u>coordinates on U.</u>

We will say that a function $(X, \mathcal{G}) \xrightarrow{\varphi} (Y, \mathcal{H})$ between two loaded spaces is <u>loaded at</u> $x \in X$, or <u>smooth at</u> x if it is continuous at x and satisfies condition (2) of the definition of loaded map for x.

<u>Lemma.</u> If $(X, \mathcal{G}) \xrightarrow{h} (Y, \mathcal{H})$ is any function between $C^\infty$ manifolds, $x \in X$, and $q^1, \ldots, q^n$ are coordinates on the domain U of a chart $U \xrightarrow{\varphi} U_o \subset \mathbb{R}^n$ such that $h(x) \in U$, then h is loaded at x if and only if each $(q^i \circ h)_x \in \mathcal{G}_x$.

<u>Proof.</u> h is loaded at x if and only if $\varphi h$ is loaded at x. So if h is loaded at x, then for every $C^\infty$ function $U_o \xrightarrow{k} \mathbb{R}$, $(k\varphi h)_x \in \mathcal{G}_x$. In particular, $(q_i h)_x = (x^i \varphi h)_x \in \mathcal{G}_x$.

Conversely, if each $(q^i h)_x \in \mathcal{G}_x$, let $V \xrightarrow{\psi} V_o$ be a chart of $X$ such that $x \in V$. Then $q^i h \psi^{-1}$ must be $C^\infty$, so for any $k$ as above, $k\varphi h\psi^{-1} = k(q^1 h\psi^{-1}, \ldots, q^n h\psi^{-1})$ is the composite of $C^\infty$ functions and hence $C^\infty$. Then since $\psi$ is loaded, $k\varphi h \in \mathcal{G}_x$. This holds for all $C^\infty$ functions $k$, so $\varphi h$ and hence $h$ are loaded.

A smooth map (a $C^\infty$-map) $h: M \to N$ between $C^\infty$ manifolds is now defined to be a continuous map which is loaded at each point $x \in M$. In other words, a function $h$ is smooth if it is continuous and if it carries good germs at each point $h(x)$ of $N$ back into good germs at $x$. It follows that the composite of smooth maps is smooth.

Example. The sphere $S^n$ is an n-manifold. The usual manifold structure is a generalization to higher dimensions of the charts obtained by stereographic projection of $S^2$. However, for $n \geq 7$ there exist other manifold structures on $S^n$, giving the so-called exotic spheres. In other words, there exist two manifold structures $\mathcal{G}$ and $\mathcal{H}$ on $S^n$ such that the identity function $(S^n, \mathcal{G}) \to (S^n, \mathcal{H})$ is not smooth.

We have described a manifold as a topological space with a function $\mathcal{G}$ assigning good germs. This function may be replaced by the function $U \mapsto \mathcal{G}(U)$ described above and called a "sheaf" (more exactly, the sheaf of germs of $C^\infty$ functions. This sheaf-theoretic definition of a manifold is equivalent to a different definition by atlases (A manifold is a topological space equipped with a suitable "maximal" atlas).

<u>The product:</u> If $M$ and $N$ are $C^\infty$ manifolds, let $M \times N$ be their product as topological spaces. If $\{U_\alpha \overset{\varphi_\alpha}{\simeq} U_\alpha^o \subseteq \mathbb{R}^m\}$ and $\{V_\beta \overset{\psi_\beta}{\simeq} V_\beta^o \subseteq \mathbb{R}^n\}$ are collections of charts covering $M$ and $N$ respectively, then the $U_\alpha \times U_\beta$ cover $M$. Furthermore, $U_\alpha \times V_\beta \overset{\varphi_\alpha \times \psi_\beta}{\simeq} U_\alpha^o \times U_\beta^o$, which is open in $\mathbb{R}^m \times \mathbb{R}^n$, so each $U_\alpha \times V_\beta$ is a manifold -- if $U_\alpha \times V_\beta \supset W \xrightarrow{g} \mathbb{R}$ and $x \in W$, then $g_x \in \mathcal{G}_x$ if and only if $g(\varphi_\alpha \times \psi_\beta)^{-1}$ is $C^\infty$ at $(\varphi_\alpha \times \psi_\beta)_x$. The manifold structures coincide on the overlaps, so by the theorem $M \times N$ is a manifold -- indeed, the topology given in the theorem is the product topology. The projections

$$\begin{array}{ccccc} M & \xleftarrow{p} & M \times N & \xrightarrow{p'} & N \\ m & \leftarrow\!\sim\!\sim & (m, n) & \sim\!\sim\!\rightarrow & n \end{array}$$

are smooth maps. If $K$ is a manifold and $f, f'$ are smooth maps

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ M & & N \end{tikzcd},$$

they are in particular continuous, so since $M \times N$ is a topological product, there exists a unique continuous map $K \xrightarrow{h} M \times N$ such that $f = ph$ and $f' = p'h$. By selecting suitable charts and coordinates we can easily show that $h$ is smooth. Hence $M \times N$ is the "categorial" product.

31. <u>The Tangent Bundle.</u>

The tangent bundles defined in Part I, § 3 for open sets of $\mathbb{R}^n$, can now be defined for manifolds.

If $M$ is a $C^\infty$ n-manifold, the tangent bundle $\begin{array}{c} T.M \\ \downarrow{\scriptstyle \pi.} \\ M \end{array}$ consists of all points $\begin{array}{c} (a, \tau_a c) \\ \downarrow{\scriptstyle \pi.} \\ a \end{array}$ where $a \in M$ and $\tau_a c$ is a tangent vector at $a$.

(More precisely, $\tau_a c$ is a tangent vector at $\varphi_a$ where $a \in U \xrightarrow{\varphi} U_o \subseteq \mathbb{R}^n$ is some fixed chart of $M$.)

For each chart $U \xrightarrow{\varphi} U^o \subseteq \mathbb{R}^n$ of $M$,

$$U^o \times \mathbb{R}^n \simeq T.U \subset T.M$$

via coordinates $q^1, \ldots, q^n; \frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}$ for $T.U$. This defines a chart on $T.M$. Any two such charts are compatible since the Jacobian of a change of coordinates in $M$ is non-zero. Thus $T.M$ is a manifold. (Apply the theorem of the previous paragraph constructing a manifold from the overlapping pieces $T.U$.)

A <u>pre-bundle</u> is $\begin{array}{c} B \\ \downarrow{\scriptstyle \pi} \\ M \end{array}$ where $B$ and $M$ are $C^\infty$ manifolds, $\pi$ is smooth, and each $\pi^{-1}(m)$ is a vector space.

Example. If $U_o$ is open in $\mathbb{R}^n$, then
$$\begin{tikzcd} U_o \times \mathbb{R}^\ell \arrow[d, "\pi_o"] & (u_o, v) \arrow[d] \\ U_o & u_o \end{tikzcd}$$
with the standard vector space structure on each $\{u_o\} \times \mathbb{R}^\ell \cong \mathbb{R}^\ell$ is a pre-bundle called a special pre-bundle.

A chart of a pre-bundle
$$\begin{tikzcd} B \arrow[d, "\pi"] \\ M \end{tikzcd}$$
consists of an open set $U$ $M$ and a pre-bundle isomorphism of
$$\begin{tikzcd} \pi^{-1}(U) \arrow[d, "\pi | \pi^{-1}(U)"] \\ U \end{tikzcd}$$
with a special pre-bundle.

A vector bundle is a pre-bundle covered by charts, (i. e., the $U$'s of all possible charts cover $M$ and the $\pi^{-1}(U)$'s cover $B$.)

Note that it is not necessary to require that addition be smooth in a vector bundle, since a vector bundle is locally like a special bundle, in which addition is automatically smooth.

$T.M$ is a vector bundle with the charts described above. Similarly, we define the cotangent bundle $T^{\cdot}M$ and the bundles constructed from the various mixed tensors. We can consider each of these as a functor which takes smooth functions between manifolds into smooth functions between vector bundles.

## 32. Bump Functions and the Extension of Germs.

We could have defined manifold with "analytic", "piecewise linear" or "continuous" replacing $C^\infty$ throughout. If we did this for "continuous", we would get a topological manifold -- a topological space which locally looks like Euclidean space.

Recall that in Part I we defined a local manifold as a set $M$ with a set $\mathcal{F}$ of "smooth" functions such that $M \simeq U_0 \subseteq \mathbb{R}^n$ and $\mathcal{F}$ corresponds one-to-one to the $C^\infty$ functions on $U_0$. We could have defined a manifold in a similar manner: as a set $M$ with a set $\mathcal{F}$ of "smooth" functions which would determine both the topology of $M$ and the "good" germs on $M$ as the germs of the functions in $\mathcal{F}$. We used germs of functions defined only on open sets of $M$, but the following theorem shows that it suffices to consider only germs of functions defined "in the large" (on all of $M$).

Theorem. If $M$ is a $C^\infty$ manifold and $g_x$ is a smooth germ at $x \in M$, then there exists a $C^\infty$ function $f: M \longrightarrow \mathbb{R}$ such that $f_x = g_x$.

Before proving the theorem we will need some preliminary results.

Definition. A topological space $X$ is compact if every open covering of $X$ has a finite subcover.

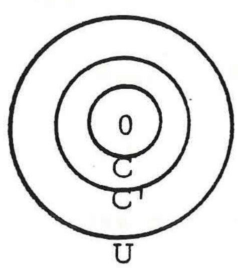
Theorem. A compact subspace of a Hausdorff topological space is closed.

(This is a standard result and can be found in any text on point-set topology.)

Lemma (Existence of "bump" functions): If $M$ is a manifold, $U \subset M$ is open, and $x \in U$, then there exist compact subsets $C$ and $C'$ of $U$ and a $C^\infty$ function $h: U \rightarrow \mathbb{R}$ such that $x \in C \subset C' \subset U$ and $h = 1$ on $C$, $h = 0$ outside $C'$.

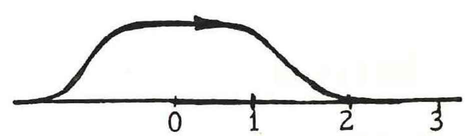
Proof of Lemma. It suffices to consider a chart containing $x$. We may take a chart in Euclidean space containing a disc about 0 of radius 3. (If not, blow up the chart by a large enough factor.)

Take $C$ = closed disc about 0 of radius 1, $C'$ = closed disc about 0 of radius 2.




If $n = 1$, define $h_o = 0$ for $x > 2$, $h_o = 1$ for $0 \leq x \leq 1$, and to be any suitable $C^\infty$ function which is 1 at 1 and 0 at 2, for $1 \leq x \leq 2$. (Problem: give an explicit formula.) Define $h_o(x) = h_o(-x)$ for $x < 0$.

For general $n$, let




$$h(x) = h(|x|^2),$$

where $|x|^2 = (\Sigma x_i^2)$ for $x = (x_1, \ldots, x_n)$.

Proof of Theorem. Let $U$ be the domain of a chart containing $x$. Construct a smooth "bump" function $b$ on $U$ by the above lemma. Let