$$f = \begin{cases} gb & \text{on } U \\ 0 & \text{outside of } U \end{cases} ,$$

Then f is continuous -- for $V \subset \mathbb{R}$ open, $f^{-1}(V) = (gb)^{-1}(V)$ is open if $0 \notin V$, and if $0 \in V$, $f^{-1}(V) = (gb)^{-1}(V) \cup (M - C')$ is open, since C' is closed by the above remark. Moreover f is smooth, since g and b are both smooth.

The theorem shows that we could have defined a manifold in terms of functions defined on the entire manifold. However, such a definition would make it more difficult to show that certain manifolds (such as tangent bundles) can be constructed by piecing together other manifolds.

Note that the Theorem would also hold for topological manifolds, but does not hold for analytic manifolds, because the bump function cannot be made analytic.

33. Volumes on Symplectic and Contact Manifolds.

Let us now review the standard set-up we use for discussing mechanics on a general differentiable manifold. Configureation space, C, is an n-dimensional manifold whose points correspond, roughly speaking, to "configurations" or "positions" of the mechanical system. Phase space is the cotangent bundle, T^*C, with the canonical 2-form ω; in local coordinates, $\omega = \Sigma\, dp_i \wedge dq^i$. We define event space to be the topological product $C \times I$, where I is an interval of time, that is, an interval on the real line with t as coordinate. A point (c, t) of event space represents the state c at the time t. Finally, state space is defined to be the product manifold $T^*C \times I$, endowed with the canonical one-form given in local coordinates as $\theta = -\Sigma\, p_i \wedge dq^i + dt$.

Now recall from Part I, § 22, that a symplectic manifold (M, ω) is a manifold M of even dimension 2n together with a closed 2-form ω such that $\omega \wedge \ldots \wedge \omega$ (n factors) is nowhere zero. Each phase space T^*C is a symplectic manifold. Similarly, a contact manifold (M, θ) is a manifold of dimension 2n + 1, where n is an integer, with a one-form θ such that the (2n+1)-form $\theta \wedge d\theta \wedge \ldots \wedge d\theta$ (n factors $d\theta$) is non-zero everywhere. State space is an example of a contact manifold. (Note: These contact manifolds are called "exact contact manifolds" in Abraham, loc. cit.)

Both a symplectic manifold and a contact manifold have a non-zero form of highest dimension; that is an "element of volume". For example, in euclidean three-space an element of volume is usually written $dx\,dy\,dz = dx \wedge dy \wedge dz$ with respect to rectangular coordinates; $r^2 \sin\theta\, dr\, d\theta\, d\varphi$ with respect to spherical coordinates, and so on. In general a volume element on an n-dimensional vector space W is a non-zero element $b \in \Lambda_n(W)$. Since the n-th exterior power $\Lambda_n(W)$ is a one-dimensional vector space, any two volume elements b and b' on W are proportional: $b' = rb$, where r is a non-zero number. Now we often speak of "right-handed" and "left-handed" coordinate systems on

Euclidean three-space; similarly, there may be two types of volume elements. To see this, say that b and b' are equivalent if the proportionality constant r is positive. This divides the volume elements up into equivalence classes: $dx \wedge dy \wedge dz = -dx \wedge dz \wedge dy = dz \wedge dx \wedge dy$, so the elements $dx \wedge dy \wedge dz$ and $dz \wedge dx \wedge dy$ are equivalent.

A volume on an n-dimensional manifold M is a form Ω on $\Omega_n(M)$ which is non-zero everywhere on M. Any two volumes Ω and Ω' are related by the formula $\Omega = f\Omega'$, where f is a smooth non-zero real-valued function on M. If f is positive everywhere, call Ω and Ω' equivalent. Then an orientation of M is defined to be an equivalence class of volumes. Since M may not have a volume in the first place, M may not be orientable; however, we have seen that symplectic manifolds and exact contact manifolds are orientable. A möbius strip is an example of a non-orientable manifold.

Let Ω be a volume on the n-dimensional manifold M. If X is a vector field on M, the Lie derivative $L_X\Omega$ is another n-form. But any two n-forms at a point are proportional. Thus there is a smooth function f such that $L_X(\Omega) = f\Omega$. We write $f = \operatorname{div} X$; notice that $\operatorname{div} X$ depends on the choice of a volume element. Does this agree with the usual notion of the divergence of a vector field? In the situation $M = \mathbb{R}^n$, with coordinates $x^1, \ldots, x^n$, we can write $\Omega = dx^1 \wedge \ldots \wedge dx^n$, and $X = \Sigma X^i \dfrac{\partial}{\partial x^i}$. Then

$$L_X\Omega = \sum_i dx^1 \wedge \dots \wedge L_X dx^i \wedge \dots \wedge dx^n .$$

But

$$L_X dx^i = \sum_j (X^j \frac{\partial}{\partial x^j})\, dx^i = \frac{\partial X^i}{\partial x^i}\, dx^i .$$

Hence

$$L_X(\Omega) = [\sum_{i=1}^{n} (\frac{\partial X^i}{\partial x^i})]\Omega$$

so $\operatorname{div} X = \Sigma \frac{\partial X^i}{\partial x^i}$, as expected.

Moreover, our generalized definition of divergence proves a suitable extension of the idea of divergence as the infinitesimal change of volume at a point. For (Part I, 24) the derivative $L_X\Omega$ describes the rate of change of the volume along the trajectories of X.

34. Poisson Brackets.

Let (M, ω) be a symplectic manifold, with symplectic coordinates $\{p_i, q^i\}$; if f and g are two real-valued functions on M, the poisson bracket of f and g with respect to the coordinates p_i, q^i is the smooth function defined by

$$\{f, g\} = \sum_{i=1}^{n} (\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}) .$$

It can be shown that the value of the poisson bracket $\{f, g\}$ of f and g does not depend on the choice of coordinates; however, we seek an invariant description of this function, since it will help us find a formulation of the laws of mechanics leading naturally to quantum mechanics.

We now develop the general algebraic machinery needed for this invariant definition of the Poisson brackets. We previously defined the exterior derivative d, which takes k-forms into $(k+1)$-forms for every non-negative integer k. Given a vector field X, there is likewise an operation i_X mapping $(k+1)$-forms ω to k-forms $i_X\omega$:

$$\begin{aligned} i_X\omega(X_1,\ldots,X_k) &= (k+1)\,\omega(X,X_1,\ldots,X_k) \\ &= \sum_i (-1)^i\, \omega(X_1,\ldots,X_{i-1},X,X_i,\ldots,X_k). \end{aligned}$$

Finally, the Lie derivative L_X takes k-forms to k-forms. We can now state three identities:

(1) $\quad i_X d + d\, i_X = L_X \qquad$ ("homotopy identity");

(2) $\quad L_X(\eta(X_1,\ldots,X_k)) = (L_X\eta)(X_1,\ldots,X_k) + \sum_{i=1}^{n} \eta(X_1,\ldots,L_X X_i,\ldots,X_k),$

where η is a k-form, so that $\eta(X_1,\ldots,X_k)$ is a function on M;

(3) $\quad 2\, d\omega(X,Y) = L_X\omega(Y) - L_Y\omega(X) - \omega([X,Y]),$

where ω is a one-form.

<u>Proof.</u> For (1), we first notice that i_X is an antiderivation: that is if α is a k-form,

$$i_X(\alpha\wedge\beta) = (i_X\alpha)\beta + (-1)^k\alpha\wedge(i_X\beta).$$

This is an easy computation from the definition of i_X.

Now we prove $i_X d\alpha + d i_X\alpha = L_X\alpha$ by induction on k: for a function f (a 0-form), $i_X f$ is defined to be zero, and we have $i_X df = \langle df, X\rangle = L_X f$. Assuming the result true for k-forms, write a general $(k+1)$-form, α, as $\Sigma\, df_i\wedge\omega_i$; by linearity it will suffice to prove the result for each summand.

But

$$L_X(df\wedge\omega) = (L_X df)\wedge\omega + df\wedge(L_X\omega),$$

while

$$\begin{aligned} & i_X d(df\wedge\omega) + di_X(df\wedge\omega) \\ & = -i_X(df\wedge d\omega) + d(i_X df\wedge\omega - df\wedge i_X\omega) \\ & = -(i_X df)\wedge d\omega + df\wedge(i_X d\omega) \\ & \qquad + (di_X df)\wedge\omega + (i_X df)\wedge d\omega + df\wedge(di_X\omega). \end{aligned}$$

Here the first and fourth terms cancel, giving

$$\begin{aligned} & df\wedge(i_X d\omega) + (di_X df)\wedge\omega + df\wedge(di_X\omega) \\ & = df\wedge L_X\omega + (di_X df\wedge\omega) \qquad \text{(by inductive assumption)} \\ & = df\wedge L_X\omega + (d(L_X f)\wedge\omega) = df\wedge L_X\omega + (L_X df)\wedge\omega. \end{aligned}$$

This proves (1).

For part (2), recall that L_X commutes with contractions, while evaluation of η at $(X_1, \ldots, X_k)$ is nothing but the contraction $\delta(\eta\otimes X_1\otimes\ldots\otimes X_k)$. With this observation (2) follows from the fact that L_X is a derivation. To derive (3), we verify the formula for $\omega = g\,dq$, since we can then extend to a general one-form ω by linearity. But since $d\omega = dg\wedge df$, it is easy to show that both sides of (3) reduce to $(L_X g)(L_Y f) - (L_X f)(L_Y g)$. A similar argument shows that for ω any k-form,

$$(4)\quad (k+1)d\omega(X_o, \ldots, X_k) = \sum_{i=0}^{k}(-1)^i L_{X_i}(\omega(X_o, \ldots, \hat{X}_i, \ldots, X_k)) + \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([X_i, X_j], X_o, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).$$

(Here the $\wedge$ over X_i means "omit X_i".)

Now let (M, ω) be a symplectic manifold. ω induces linear mappings (§21) $X \to X^b$ and $\alpha \to \alpha^\#$ taking vector fields into co-vector fields (= 1-forms), and vice versa. We may now define the poisson bracket of two one-forms, α and β, by

Definition. $\{\alpha, \beta\} = -[\alpha^\#, \beta^\#]^b$.

In other words, we turn the forms temporarily into vector fields, take the Lie bracket, and return to the space of forms. The minus sign is chosen for convenience in proving such formulas as

Proposition. $\{\alpha, \beta\} = -L_{\alpha^\#}\beta + L_{\beta^\#}\alpha + d(i_{\alpha^\#} L_{\beta^\#}\omega)$.

Proof. ω is closed, hence by (4) above

$$0 = 3d\omega(X, Y, Z) = L_X(\omega(Y, Z)) + L_Y(\omega(Z, X)) + L_Z(\omega(X, Y))$$
$$- \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X).$$

If we let $X = \alpha^\#$, $Y = \beta^\#$, and recall that, by the definition of $\#$, we have $\omega(\alpha^\#, Y) = \frac{1}{2}\alpha(Y)$, the above equation becomes

$$0 = L_{\alpha^\#}(\tfrac{1}{2}\beta(Z)) - L_{\beta^\#}(\tfrac{1}{2}\alpha(Z)) + L_Z(\omega(\alpha^\#, \beta^\#))$$
$$+ \omega(\{\alpha, \beta\}^\#, Z) + \omega([\alpha^\#, Z], \beta^\#) - \omega([\beta^\#, Z], \alpha^\#).$$

Therefore

$$\begin{aligned} -L_{\alpha^\#}(\tfrac{1}{2}\beta(Z)) + L_{\beta^\#}(\tfrac{1}{2}\alpha(Z)) - L_Z(\omega(\alpha^\#, \beta^\#)) \\ = \tfrac{1}{2}(\{\alpha, \beta\})Z - \tfrac{1}{2}\beta[\alpha^\#, Z] + \tfrac{1}{2}\alpha[\beta^\#, Z] \\ = \tfrac{1}{2}(\{\alpha, \beta\})Z - \tfrac{1}{2}\beta L_{\alpha^\#}Z + \tfrac{1}{2}\alpha L_{\beta^\#}Z. \end{aligned}$$

Now the proposition follows from the three identities

$$L_{\alpha^\#}(\tfrac{1}{2}\beta(Z)) = (\tfrac{1}{2}L_{\alpha^\#}\beta)Z + \tfrac{1}{2}\beta(L_{\alpha^\#}Z)$$

$$-L_{\beta^\#}(\tfrac{1}{2}\alpha(Z)) = -\tfrac{1}{2}(L_{\beta^\#}\alpha)Z - \tfrac{1}{2}\alpha(L_{\beta^\#}Z)$$

$$-2L_Z(\omega[\alpha^\#, \beta^\#]) = d(i_{\alpha^\#}i_{\beta^\#}\omega)Z,$$

of which the first two are merely (2), above, and the third is a consequence of the equation $i_X\omega(Y) = 2\omega(X, Y)$.

Corollary 1. If β is closed then $\{\alpha, \beta\} = L_{\beta^\#}\alpha$.

Proof. By the homotopy identity,

$$\begin{aligned} L_{\alpha^\#}\beta &= i_{\alpha^\#}d\beta + di_{\alpha^\#}\beta \\ &= 0 + 2d(\tfrac{1}{2}\beta(\alpha^\#)) = 2d(\omega(\beta^\#, \alpha^\#)) = d(i_{\alpha^\#}i_{\beta^\#}\omega) \end{aligned}$$

Now use the proposition.

Corollary 2. If α and β are closed, $\{\alpha, \beta\} = L_{\beta^\#}\alpha = -L_{\alpha^\#}\beta = 2d(\omega(\beta^\#, \alpha^\#))$.

Corollary 3. If α and β are closed, $\{\alpha, \beta\}$ is exact.

For $\{\alpha, \beta\} = d(2\omega(\beta^\#, \alpha^\#))$.

Now by using #, we see that each function f on M determines a vector field $X_f = (df)^\#$.

Corollary 4. If f and g are smooth functions on M, then

$$\begin{aligned} \{df, dg\} &= L_{X_g}(df) = d(L_{X_g}f) \\ &= -L_{X_f}(dg) = -d(L_{X_f}g) \\ &= 2d(\omega(X_g, X_f)) . \end{aligned}$$

Definition. The Poisson bracket of the functions f and g is

$$\{f, g\} = L_{X_g} f \quad (X_g = (dg)^{\#}). \text{ (Hence } d\{f, g\} = \{df, dg\}).$$

Proposition. $L_{X_g} f = -L_{X_f} g = -2\omega(X_g, X_f)$.

Proof.

$$\begin{aligned} L_{X_g} f &= \langle df, X_g\rangle = \langle df, dg^{\#}\rangle = 2\omega(df^{\#}, dg^{\#}) \\ &= 2\omega(X_f, X_g) = -2\omega(X_g, X_f) = -2\omega(dg^{\#}, df^{\#}) \\ &= -\langle dg, df^{\#}\rangle \\ &= -L_{X_f} g. \end{aligned}$$

In particular, $L_{X_g} f = 0$ if and only if $L_{X_f} g = 0$. Thus f is constant on the trajectories of g if and only if g is constant on the trajectories of f. (By the trajectories of f we mean those of the vector field X_f.)

Of course, we must check that this definition agrees with our coordinate-wise notion of poisson bracket. Let the symplectic coordinates be $\{p_i, q^i\}$. This means that $\omega = \Sigma\, dp_i \wedge dq^i$. Any 1-form α can be written

$$\alpha = \Sigma\, h_i dq^i + \Sigma\, k^j dp_j ,$$

while any vector field X can be written

$$X = \Sigma\, X^i \frac{\partial}{\partial q^i} + \Sigma\, T^j \frac{\partial}{\partial p_j} .$$

Then we have seen that

$$\begin{aligned} X^{\flat} &= \Sigma\, T^i dq^i - \Sigma\, X^i dp_i \\ \alpha^{\#} &= -\Sigma\, k^i \frac{\partial}{\partial q^i} + \Sigma\, h_i \frac{\partial}{\partial p_i} . \end{aligned}$$

Thus $\{f, g\} = -L_{X_f} g = \Sigma\left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j}\right)$ as expected.

Since our poisson bracket $\{\ ,\ \}$ was defined invariantly from the 2-form ω, the formula holds for any symplectic (= canonical) coordinates. In particular, this formula gives the poisson bracket of any two coordinate functions. We can deduce hence that a set of 2n functions $Q^1, \ldots, Q^n, P_1, \ldots, P_n$ on a symplectic manifold are symplectic coordinates if and only if they satisfy the relations

$$\{P_i, Q_j\} = \delta_{ij}$$
$$\{P_i, P_j\} = \{Q_i, Q_j\} = 0$$

for all i and j.

One can also prove that a transformation $\varphi: (M, \omega) \longrightarrow (M, \omega)$ is symplectic if and only if it preserves all poisson brackets of functions.

Proposition. For any three smooth functions on a symplectic manifold

$$\{f, \{g, h\}\} + \{g\{h, f\}\} + \{h, \{f, g\}\} = 0.$$

This asserts that the set of smooth functions is a Lie algebra under the poisson bracket $\{\ ,\ \}$.

Proof.

$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$

$$= L_{X_f} L_{X_g} h - L_{X_g} L_{X_f} h + L_{X_{\{f, g\}}} h$$

and this is zero, because

$$X_{\{f, g\}} = (d\{f, g\})^{\#} = \{df, dg\}^{\#} = -\{df^{\#}, dg^{\#}\} = -[X_f, X_g].$$

Here is an application of the antisymmetry of the poisson bracket. Consider k particles moving in three dimensional space. Their position is then specified by 3k coordinates

$$(x_1, y_1, z_1; x_2, y_2, z_2; \ldots, x_k, y_k, z_k),$$

so that the configuration space C is $\mathbb{R}^{3k}$. In the corresponding phase space $M = T^* C$ we can write down the Hamiltonian function H in terms of the potential energy V and the usual kinetic energy of the 3k particles. If we assume that V depends only on the distances between particles, then the Hamiltonian H is left fixed by the transformations of M induced by rigid motions, like translations and rotations, in $\mathbb{R}^3$. Let X_g be the vector field corresponding to such a translation; then X_g leaves H invariant. By anti-symmetry, X_H must leave g invariant; that is, since the system moves along the trajectories of H, g is a constant of the motion. For translations, g turns out to be the linear momentum, while for rotations g is angular momentum. We have just derived the familiar conservation-of-momentum laws. In general, any function f with $\{f, H\} = 0$ is a constant of the motion.

35. <u>Submanifolds and Immersions.</u>

We will study "energy surfaces" (submanifolds of constant energy); for this we need some facts about submanifolds. In a number of places in these lectures we have used (and will be using) the theorem below and its corollaries. (Here $Df(m)$ is the map induced by f on the tangent space at the point m.)

<u>Theorem</u>. (Inverse Function Theorem): Let $M \xrightarrow{f} N$ be a smooth function. If $Df(m)$ is an isomorphism, then f is a local diffeomorphism at m; i.e., there are neighborhoods U of m and V of fm such that $f(U) = V$ and $f|U: U \longrightarrow V$ has a smooth inverse.

<u>Corollary 1.</u> (Implicit Function Theorem): Let $M \xrightarrow{f} N$ be a smooth function. If $Df(m)$ is a surjection, then f is locally a projection; i.e., there are charts (U, ϕ) at m and (V, ψ) at fm such that

$\phi U = U' \times V'$

$\psi V = V'$

and $\psi \circ f \circ \phi^{-1}$ is the projection of $U' \times V'$ onto V'.

$$\begin{tikzcd}
M \arrow[rr, "f"] & & N \\
m \in U \arrow[u, phantom, "\cup"] \arrow[d, "\phi", "\cong"'] & & fm \in V \arrow[u, phantom, "\cup"] \arrow[d, "\psi", "\cong"'] \\
U' \times V' \arrow[rr, "\psi \circ f \circ \phi^{-1}"] & & V'
\end{tikzcd}$$

<u>Corollary 2.</u> Let $M \xrightarrow{f} N$ be a smooth function. If $Df(m)$ is an injection, then f is locally an injection; i.e., there are charts (U, ϕ) at m and (V, ψ) at fm such that $\phi U = U'$, $\psi V = U' \times V'$, and $\psi \circ f \circ \phi^{-1}$ is the injection of U' into $U' \times V'$ as $U' \times 0$.

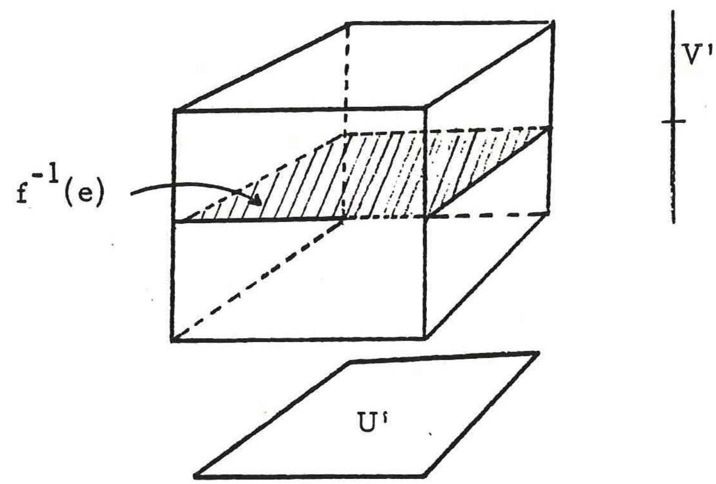
In what follows, M and N are manifolds and $a \in M$. An <u>immersion</u> of M in N is a smooth function $M \xrightarrow{f} N$ with the property: for each $f(a) \in N$, there is a chart at $f(a)$ with coordinates $q^1, \ldots, q^n$ such that $q^1 f, \ldots, q^d f$ are coordinates for a chart at a, for some $d \leq n$.

By Corollary 2 above, a smooth function $M \xrightarrow{f} N$ is an immersion if and only if $Df(a)$ is an injection for every $a \in M$. An <u>embedding</u> is an immersion which is a homeomorphism onto its image endowed with the subspace topology. A weaker notion of embedding which sometimes is used is an immersion that is an injection (1-1 function); but the stronger sense seems to be what we want for mechanics. If $M \subset N$ and the inclusion is an embedding, then M is a <u>submanifold</u> of N. One last definition: the point $e \in N$ is a <u>regular value</u> of f if and only if the Jacobian $Df(a)$ has maximum rank for every a such that $f(a) = e$. Since $Df(a)$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ it will have maximum rank when it is surjective if $m \geq n$ and when it is injective if $m \leq n$.

<u>Theorem</u>. Let N, P be manifolds and $N \xrightarrow{f} P$ a smooth function. If $e \in P$ is a regular value of f, then $f^{-1}(e)$ is a submanifold of N.

<u>Proof</u>. Since the inclusion $f^{-1}(e) \subset \overset{N}{\cancel{M}}$ is clearly a homeomorphism onto its image we need only show the immersion property. Take $a \in N$ so that $f(a) = e$.

Case (i). Df(a) is surjective: Then Corollary 1 of the Inverse Function Theorem gives a chart (U, ϕ) at a such that $\phi(f^{-1}(e) \cap U) = U'$ and $\phi U = U' \times V'$. Locally the picture shows what the manifold M looks like near a.



Case (ii). Df(a) is injective: Then f is an injection near a, so $f^{-1}(e)$ is a set of isolated points each of which is a submanifold. Thus the union of these is a submanifold, so the theorem is proved.

If $P = \mathbb{R}$ in the theorem, it is useful to have explicit local coordinates for a point a such that $f(a) = e$. Let $q^1, \ldots, q^n$ be any coordinates around a; then $Df(a) = \left(\frac{\partial f}{\partial q^1}, \ldots, \frac{\partial f}{\partial q^n}\right)_a$ where e is a regular value means one of the entries is non-zero, say $\frac{\partial f}{\partial q^n}$. The change of coordinates
$$\begin{aligned} q^i &\longrightarrow q^i \quad 1 \le i \le n-1 \\ q^n &\longrightarrow f \end{aligned}$$
has Jacobian matrix

$$\begin{pmatrix} 1 & & & O \\ & 1 & & \\ & & \ddots & \\ O & & & \dfrac{\partial f}{\partial q^n} \end{pmatrix}$$

so $q^1, \ldots, q^{n-1}, f$ are coordinates around a in N such that $q^1, \ldots, q^{n-1}$ are coordinates around a in $f^{-1}(e)$.

36. Invariants on a Symplectic Manifold.

We first study quantities invariant under a vector field on any manifold.

Definition. Let M be a manifold and X a vector field on M; then a k-form α is invariant under X if and only if $L_X\alpha = 0$.

We have the equivalences

$$L_X\alpha = 0 \text{ iff } F_t^*\alpha = \alpha, \text{ where } F \text{ is a flow of } X,$$
$$\text{iff } \alpha \text{ is constant on integral curves of } X.$$

The following properties are easy to prove.

(i) α invariant under X implies $i_X\alpha$ and $d\alpha$ are invariant under X.

(ii) α and β invariant under X implies $\alpha \wedge \beta$ is invariant under X.

(iii) α invariant under X and $L_X Y = 0$ implies $i_Y\alpha$ is invariant under X,

where X, Y are vector fields and α, β forms.

Proof of (i) for $i_X\alpha$: By the "homotopy identity" (1) of 33, $L_X = i_X d + d i_X$, thus

$$L_X(i_X\alpha) = i_X d\, i_X\alpha + d\, i_X i_X\alpha = -i_X i_X d\alpha + d\, i_X i_X\alpha.$$

But the definition of i_X shows that $i_X i_X = 0$, since we are dealing with alternating tensors, so $L_X(i_X\alpha) = 0$.

Proof of (ii): L_X is a derivation.

Proof of (iii):

$$\begin{aligned} L_X(i_Y\alpha) &= L_X(\mathcal{C}(\alpha\otimes Y)) \\ &= \mathcal{C}(L_X(\alpha\otimes Y)) \\ &= \mathcal{C}(L_X\alpha\otimes Y + \alpha\otimes L_X Y) = 0 \end{aligned}$$

where $\mathcal{C}$ is the contraction operator which commutes with Lie derivatives.

A submanifold V of M is an <u>invariant submanifold of X</u> if for each $a \in V$, $X_a \in T_aV \subset T_aM$.

<u>Theorem.</u> Suppose M a manifold and X a vector field on M. If the function $M \longrightarrow \mathbb{R}$, as a 0-form, is invariant under X and if e is a regular value of k, then $k^{-1}(e)$ is an invariant submanifold of X.

<u>Proof.</u> By the previous theorem, $k^{-1}(e)$ is a submanifold of M and for each $a \in M$ such that $k(a) = e$ there are coordinates $q^1,\ldots,q^m$ around a in M with $q^1,\ldots,q^{m-1}$ coordinates for a in $k^{-1}(e)$ and $\frac{\partial k}{\partial q^m} \neq 0$. By definition of L_X for $X = \Sigma X^i \frac{\partial}{\partial q^i}$,

$$0 = L_X k = \langle dk, X\rangle = \sum_{i=1}^{m-1} \frac{\partial k}{\partial q^i} X^i + \frac{\partial k}{\partial q^m} X^m = \frac{\partial k}{\partial q^m} X^m$$

at a because k is constant on $k^{-1}(e)$. Thus $X^m a = 0$; but

$T_a(k^{-1}(e)) = \{w \in T_aM \mid \text{the last component } w_m = 0\}$. So $X_a \in T_a(k^{-1}(e))$, which was to be proved.

Proposition. If the function $M \xrightarrow{k} \mathbb{R}$ is an invariant of the vector field X on the manifold M and if e is a regular value of K, then a trajectory of X which meets a connected component Σ_e of $k^{-1}(e)$ lies entirely in Σ_e.

(Here connected means path connected; i. e., any two points of Σ_e can be connected by a path lying entirely in Σ_e.)

Proof. Let γ be a trajectory for X starting at the point $a \in \Sigma_e$. Because Σ_e is an invariant submanifold of X, $X|\Sigma_e$ is a vector field on Σ_e. The existence and uniqueness theorems for differential equations say there is a unique trajectory γ' in Σ_e starting at a which satisfies the differential equation for $X|\Sigma_e$. But γ is such a trajectory, thus $\gamma = \gamma'$ and γ lies in Σ_e.

Now consider invariants for a symplectic manifold (M, ω) of dimension $2n$.

Definition. A vector field X on (M, ω) is locally Hamiltonian if and only if ω is invariant under X; i. e., $L_X\omega = 0$. Equivalent conditions are

(a) $di_X\omega = 0$,

(b) $X^\flat$ is closed,

(c) $X^\flat = dH$ locally, for some function H.

The vector field **X** is globally Hamiltonian if and only if there is a smooth $M \xrightarrow{H} \mathbb{R}$ such that $X = (dH)^{\#}$, or equivalently, $X^{\flat}$ is exact.

Recall that a volume on M is a nowhere zero 2n-form. For example $\Omega = \frac{-1^{n/2}}{n!} \omega \wedge \ldots \wedge \omega$ (n times) is a volume. If X is a locally Hamiltonian vector field, then $L_X \omega = 0$, so $L_X \Omega = 0$. The volume element thus is constant under motion along **X**. This statement becomes Liouville's Theorem when translated into the language of statistical mechanics. In detail, in statistical mechanics a system of n particles is replaced by a single particle in 3n-dimensional configuration space, and hence by a point moving in 6n-dimensional phase space. An ensemble of systems thus corresponds to an ensemble of points in phase space. Liouville's Theorem states that the density of this ensemble is constant along the trajectories.

37. Submanifolds of Constant Energy.

The last proposition shows that for the trajectories defined by the globally Hamiltonian vector field $(dH)^{\#}$ it is appropriate to restrict consideration to the submanifolds where H is constant. We now examine the structure of such submanifolds for any suitable function K.

<u>Theorem.</u> (Hamilton-Jacobi): Let X be a vector field on an m-dimensional manifold M, $M \xrightarrow{K} \mathbb{R}$ an invariant of X, e a regular value of K, and V a connected component of $K^{-1}(e)$; then

1° V is an embedded submanifold of dimension $m-1$.

2° If M is oriented, so is V.

3° If M has an X-invariant volume Ω, then V has an $X|V$-invariant volume Ω_V.

<u>Proof.</u> 1° has already been proven.

2°. We will need a "normal" vector field N for V; that is, a smooth function $V \longrightarrow T.M$ with the property that for each $a \in V$

$$T_{\nu a}M = \nu_*(T_aV) \oplus \mathbb{R}\cdot Na$$

where ν is the inclusion of V in M. To get such an N let g be any Riemannian metric on M (see the end of the proof for a different method), and set $N = g_{\#} \circ dK \circ \nu$. Thus Na for each point a is the unique vector such that $g(Na, -) = d_{\nu a}K$ is the differential of K at νa.

$$\begin{tikzcd}
 & T.M \arrow[d, "\pi."] & & T^{*}M \arrow[lll, "g_{\#}"'] \\
V \arrow[ur, dashed, "N"] \arrow[r, "\nu"'] & M \arrow[urr, "dK"'] & &
\end{tikzcd}$$

To prove that N is a "normal" vector field it suffices to show that $N_a \notin \nu_*(T_aV)$. Now

$$\nu_*(T_aV) = \{Y \in T_{\nu a}M \mid \langle d_{\nu a}K, Y\rangle = 0\},$$

but $\langle d_{\nu a}K, Na\rangle = g(Na, Na)$ which is not zero unless $Na = 0$. Since e is a regular value $d_{\nu a}K = g(Na, -) \neq 0$, so Na cannot be zero.

Suppose Ω is a non-zero volume on M, then $\nu^*(i_N\Omega)$ is an $(m-1)$-form on V and

$$\begin{aligned}\nu^*(i_N\Omega)(w_1, \dots, w_{m-1}) &= i_N\Omega(\nu_* w_1, \dots, \nu_* w_{m-1}) \\ &= m\Omega(N, \nu_* w_1, \dots, \nu_* w_{m-1}) \\ &\neq 0.\end{aligned}$$

Therefore, $\nu^*(i_N\Omega)$ is a volume on V. Consequently, V is orientable. This applies in particular when M is symplectic, and hence orientable.

3° First we need a lemma:

Lemma. If γ is an $(m-1)$-form on M, $a \in V$, and $w_1, \dots, w_{m-1} \in T_aV$, then

$$\begin{aligned}(dK \wedge \gamma)_a&(Na, \nu_* w_1, \dots, \nu_* w_{m-1}) \\ &= c \langle dK, N \rangle_a \gamma a(\nu_* w_1, \dots, \nu_* w_{m-1})\end{aligned}$$

with c a non-zero constant.

<u>Proof of Lemma.</u> We use the definition of a form as an alternating tensor. Indeed

$$(dK \otimes \gamma)_a(Na, \nu_* w_1, \dots, \nu_* w_{m-1}) = \langle dK, N \rangle_a \gamma_a(\nu_* w_1, \dots, \nu_* w_{m-1}),$$

so $(dK \wedge \gamma)_a(Na, \nu_* w_1, \dots, \nu_* w_{m-1})$

$$\begin{aligned}&= \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^{\sigma} (dK \otimes \gamma)_a (Na, \nu_* w_1, \dots, \nu_* w_{m-1}) \\ &= \frac{1}{m} \langle dK, N \rangle_a \gamma_a(\nu_* w_1, \dots, \nu_* w_{m-1}),\end{aligned}$$

because $\langle dK, w_i \rangle_a = 0$ and γ is a form. Putting $c = \frac{1}{m}$ proves the lemma.

Now assume Ω is an X-invariant volume on M and let $\theta = i_N\Omega$. Then $dK\wedge\theta$ is an m-form, and hence is a multiple of Ω:

$$dK\wedge\theta = h\cdot\Omega \qquad \text{where } h: M \longrightarrow \mathbb{R} \text{ is smooth.}$$

In order to make $\nu^*\theta$ invariant, we want $h = 1$ -- if $h \neq 0$, we can multiply by h^{-1}. But

$$\begin{aligned} h a\Omega_a & (Na, \nu_* w_1, \ldots, \nu_* w_{m-1}) \\ &= (dK\wedge\theta)_a (Na, \nu_* w_1, \ldots, \nu_* w_{m-1}) \\ &= c\langle dK, N\rangle_a \theta_a(\nu_* w_1, \ldots, \nu_* w_{m-1}) \\ &= c'\langle dK, N\rangle_a \Omega_a(Na, \nu_* w_1, \ldots, \nu_* w_{m-1}). \end{aligned}$$

For $w_1, \ldots, w_{m-1}$ linearly independent in T_aV, all the terms, except ha, at both ends of this equation are non-zero; therefore ha is also non-zero. Thus set $\bar\theta = h^{-1}\theta$; then $dK\wedge\bar\theta = \Omega$ and $\nu^*\bar\theta$ is a volume (as in 2^o). We must finally show $\nu^*\bar\theta$ invariant under $XN = \nu^*X$. First

$$0 = L_X\Omega = L_X dK\wedge\bar\theta + dK\wedge L_X\bar\theta$$

so $dK\wedge L_X\bar\theta = 0$.

Using the lemma again

$$\begin{aligned} 0 &= (dK\wedge L_X\bar\theta)_a (Na, \nu_* w_1, \ldots, \nu_* w_{m-1}) \\ &= c\langle dK, N\rangle_a (L_X\bar\theta)_a (\nu_* w_1, \ldots, \nu_* w_{m-1}) \\ &= c\langle dK, N\rangle_a \nu^*(L_X\bar\theta)_a (w_1, \ldots, w_{m-1}) \\ &= c\langle dK, N\rangle_a (L_{X|V}\nu^*\bar\theta)_a (w_1, \ldots, w_{m-1}). \end{aligned}$$

Since the first two factors c and $\langle dK, N\rangle_a$ on the right are non-zero, we get $L_{X|V}(\nu^*\bar\theta) = 0$, which proves 3^o.

This proof of the Hamilton-Jacobi theorem is a presentation of that given in Abraham, loc. cit., in numbers 11.11, 11.15, 15.13, 16.27.

This proof made use of the fact that there exists a Riemannian metric on any manifold. The use of this result is efficient and suggestive, but there is no unique or canonical such metric. After the lecture, Alan Waterman suggested the following (standard) way of avoiding the choice of a metric.

Lemma. For M, K, and V as in the theorem, let Ω be any m-form on M. Then there is a 1-form θ on V such that for any point a of V there is a coordinate neighborhood of a in M and an (m-1)-form β on the neighborhood with

$$\theta = \nu^* \beta \ , \quad \Omega = \beta \wedge dK.$$

Proof. Let $\nu: V \rightarrow M$ be the inclusion. Since $dK \neq 0$ at e, we can take K as one of the coordinates at a in M. The form Ω of maximal dimension can then be written locally as $\Omega = \beta \wedge dK$, where β is some (m-1)-form on M. This form β is not unique, but if also $\Omega = \beta' \wedge dK$ in the same coordinate neighborhood, then a representation with coordinates shows that $\nu^* \beta = \nu^* \beta'$. Therefore $\theta = \nu^* \beta = \nu^* \beta'$ is an (m-1)-form well-defined everywhere on V, and the lemma holds.

The theorem itself is now readily proved from this lemma; in particular, since Ω and K are both invariants for X, it follows that θ is an $X|V$-invariant volume.

Chapter V. QUALITATIVE PROPERTIES OF VECTOR FIELDS

38. Orbits.

This chapter is devoted to the study of the qualitative properties of the trajectories (integral curves) of vector fields on manifolds. Except for the first section, the chapter consists of notes of lectures given by Prof. René Thom of the Institute des Hautes Etudes Scientifiques (France).

Hence suppose M is a manifold and X a vector field on M. Consider trajectories of X; that is, the curves $c: I \longrightarrow M$ such that the tangent vector to c at each point is the value of X at that point. We shall assume that c and the given interval I are chosen so that $0 \in I$.

Let

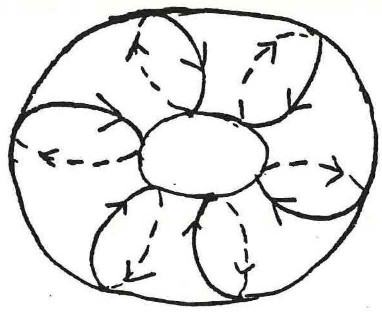
$$\mathcal{D}_X = \{(a,t) \mid a \in M, t \in \mathbb{R} \text{ and there exists a trajectory } c: I \longrightarrow M \text{ of } X \text{ with } c(0) = a \text{ and } t \in I\}.$$

By the uniqueness theorem for differential equations, there exists a smooth map (a "flow")

$$F: \mathcal{D}_X \longrightarrow M$$

such that for each a, $F(a, -)$ is a trajectory for X -- in fact is the maximal trajectory through a. ($\mathcal{D}_X$ is an open subset of $M \times \mathbb{R}$, so it is meaningful to require that F be smooth.)

An orbit of X is the image in M of a maximal trajectory. A closed orbit of X is an orbit which is compact. For example, the vector field going around the torus as indicated has (infinitely) many closed orbits.



Finally, the support of X is the closure of the set $\{a \in M \mid X(a) \neq 0\}$.

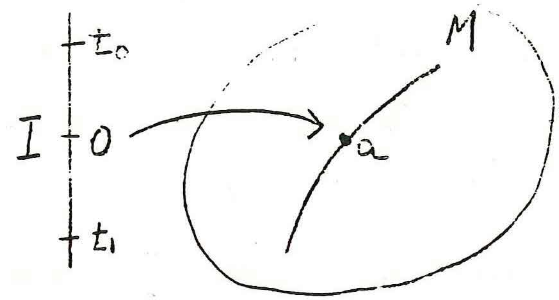
A vector field X is complete when $\mathcal{D}_X = M \times \mathbb{R}$. It is equivalent to say that every integral curve of X can be extended to one whose domain of definition is the whole real line. This cannot always happen. For example, let U be the first quadrant of $\mathbb{R}^2$, with the usual coordinates x, y. Then the vector field $X(x, y) = (\frac{1}{x}, 1)$ is not complete. (Verification is left to the reader.) If however, the closure of the set of points p where $X(p) \neq 0$ is compact, then X is complete.

The most important thing about a complete vector field is that it yields a one-parameter group of diffeomorphisms. For each $s \in R$, there is a diffeomorphism Φ_s of the manifold such that $\Phi_o = 1$ and $\Phi_s \Phi_t = \Phi_{s+t}$. Explicitly, $\Phi_s(p)$ is the value of the integral curve of X with initial conditions p at time s. In contrast a non-complete vector field yields only local diffeomorphisms rather than global ones.

Theorem. Let X be a vector field on M. If M is a compact manifold or if X has compact support, then X is complete.

We sketch the proof in the case that M is compact. We shall use the fact that in a compact space every infinite sequence of points has a convergent subsequence.

We want to show that we can prolong any trajectory $c: I \longrightarrow M$ where $I = (t_o, t_1)$ for $t_o < 0 < t_1$. Suppose c were maximal and $t_1 < \infty$. Let $\{t_i\}$ be a sequence of points in I convergent to t_1. Then the sequence $\{c(t_i)\}$ in M has a limit point m (i.e., some subsequence of $\{c(t_i)\}$ converges to m). Apply the existence theorem of differential equations to get a flow box $F: U \times I_1 \longrightarrow M$ at m. Thus $U \subset M$ is an open set containing m. But m must be in the closure of the image of c. Therefore, U contains some point $c(\bar{t})$ in the image of c. $F(c(\bar{t}), -)$ is a trajectory through $c(\bar{t})$ which must extend c.



A similar argument applies to the case when M is not compact but X has compact support.

Suppose next that K is a compact orbit of X. Then we may as well assume that K is not a point, and consequently that X is never zero on K. The main result is that compact orbits are periodic.

<u>Theorem</u>. Suppose $\varphi: \mathbb{R} \longrightarrow M$ is a non-constant integral curve of X with compact image K. Then there is a $\tau > 0$ such that $\varphi(t+\tau) = \varphi(t)$, all t.

The least such t is called the <u>period</u> of the orbit.

Proof. We first notice that we can assume the vector field X to be complete. For if we take a neighborhood W of K with compact closure and a C^∞ function α which is 1 near K and 0 off W, then αX is a vector field with compact support which agrees with X near K; also φ is an integral curve of αX.

It suffices to show that there are points $t \neq t'$ such that $\varphi(t) = \varphi(t')$. For if $\{\Phi_s : M \longrightarrow M\}$ is the corresponding 1-parameter group of diffeomorphisms of M, it then follows that

$$\varphi(s + t - t') = \Phi_{s-t'}\varphi(t) = \Phi_{s-t'}\varphi(t') = \varphi(s)$$

for all $s \in R$. If $t > t'$, this is the conclusion $(\tau = t - t')$ of the theorem; since t and t' are symmetric, we may suppose this is so.

We thus want to prove the following

Lemma. There are points $t \neq t'$ such that $\varphi(t) = \varphi(t')$.

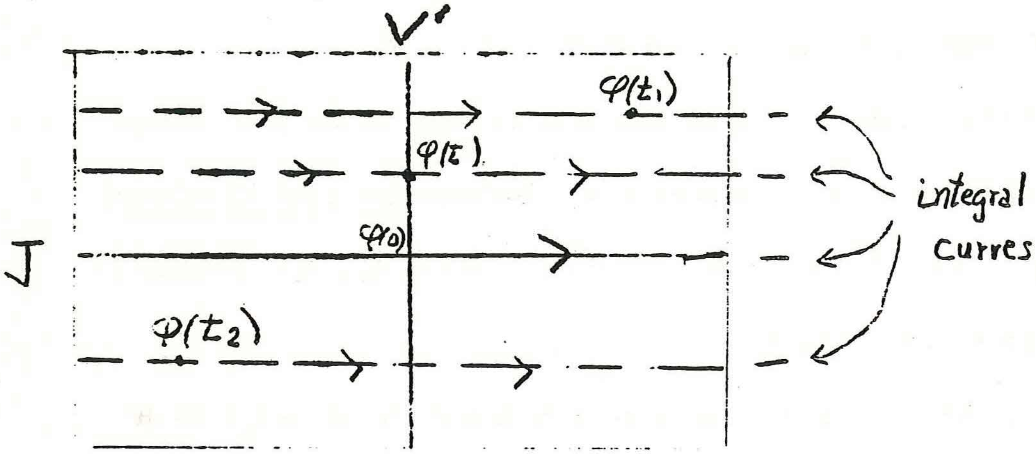
Proof As noted earlier, X is non-zero on a neighborhood of K. Therefore, by the inverse function theorem, there exists an open interval $J \subset \mathbb{R}$ containing 0 and an open disk V' of radius ε such that $J \times V'$ is a coordinate neighborhood at $X(0)$ with coordinates t and $v^1, \ldots, v^{n-1}$, and such that locally

$$\varphi(t) = (t, 0).$$

By continuity we may assume that if

$$X = \sum w_i \frac{\partial}{\partial v^i} + y \frac{\partial}{\partial t}$$

then y is bounded away from zero (and in fact is positive on $J \times V'$).



Suppose $\varphi(t) \in J \times V'$. By our hypothesis that y is positive, a point can travel backward or forward along the trajectory of φ until it meets the disk $0 \times V'$. (In the above picture, the point travels backwards from $\varphi(t_1)$ and forwards from $\varphi(t_2)$.)

Again by our hypotheses on X, if $\varphi(t) = (0, v')$, then no other nearby point can also get mapped into $0 \times V'$. This is because the component y is bounded away from zero, and hence the integral curves must have at least some fixed positive velocity in the $\frac{\partial}{\partial t}$ direction. In other words, the set of points t such that $\varphi(t) \in 0 \times V'$ is an <u>isolated</u> subset of R. But it is easy to show that any such subset is countable; that is, it is in 1-1 correspondence with a subset of the positive integers.

Since $K \cap (0 \times V')$ is countable and $[0, 1)$ is uncountable, there exists an $\varepsilon' < \varepsilon$ such that no point of K has distance exactly ε' from the origin. Let V be the closed disk of radius ε'; then $K \cap (0 \times V)$ is compact and countable, and is contained in the interior of V. As a subset of a metric space, it has a notion of distance.

It is left to the reader to prove the next result, which is a trivial consequence of Baire's theorem (see e.g., Kelley, General Topology)

Sublemma. Let Y be a compact metric space which is countable. Then Y has an isolated point.

Therefore, there is a t_o and a neighborhood U of $\varphi(t_o)$ in V such that $K \cap U = \emptyset$. Again, it follows by continuity that there is a $U' \subseteq U$ neighborhood of $\varphi(t_o)$ and a $J_o \subseteq J$ neighborhood of 0 such that $K \cap (J_o \times U') = J_o \times \varphi(t_o)$

Composing the result with the diffeomorphism Φ_s for $s = t_i - t_o$, we see that for any point $\varphi(t_i) \in K$, there exists a coordinate neighborhood $J \times U$ of $\varphi(t_i)$ such that the intersection of K with the neighborhood is $J \times 0$.

Since K is compact, there is a finite collection of $(J_i \times U_i)$ covering it. Since the inverse images of the $J_i \times 0$ then cover $\mathbb{R}$, it follows that some point t' outside some interval $t_k + J_k$ gets mapped into $J_k \times 0$. But we know some $t \in t_k + J_k$ gets mapped onto any point of $J_k \times 0$, and for these choices $\varphi(t') = \varphi(t)$.

We now want to ask what the critical elements of a vector field X are. These are of two types:

1) $a \in M$ such that $X(a) = 0$ (for example, the south pole on the sphere with a vector field which everywhere points downward). Since