

$$f = \begin{cases} gb & \text{on } U \\ 0 & \text{outside of } U \end{cases} .$$

Then f is continuous -- for $V \subset \mathbb{R}$ open, $f^{-1}(V) = (gb)^{-1}(V)$ is open if $0 \notin V$, and if $0 \in V$, $f^{-1}(V) = (gb)^{-1}(V) \cup (M - C')$ is open, since C' is closed by the above remark. Moreover f is smooth, since g and b are both smooth.

The theorem shows that we could have defined a manifold in terms of functions defined on the entire manifold. However, such a definition would make it more difficult to show that certain manifolds (such as tangent bundles) can be constructed by piecing together other manifolds.

Note that the Theorem would also hold for topological manifolds, but does not hold for analytic manifolds, because the bump function cannot be made analytic.

33. Volumes on Symplectic and Contact Manifolds.

Let us now review the standard set-up we use for discussing mechanics on a general differentiable manifold. Configuration space, C , is an n -dimensional manifold whose points correspond, roughly speaking, to "configurations" or "positions" of the mechanical system. Phase space is the cotangent bundle, T^*C , with the canonical 2-form ω ; in local coordinates, $\omega = \sum dp_i \wedge dq^i$. We define event space to be the topological product $C \times I$, where I is an interval of time, that is, an interval on the real line with t as coordinate. A point (c, t) of event

space represents the state c at the time t . Finally, state space is defined to be the product manifold $T^*C \times I$, endowed with the canonical one-form given in local coordinates as $\theta = -\sum p_i \wedge dq^i + dt$.

Now recall from Part I, §22, that a symplectic manifold (M, ω) is a manifold M of even dimension $2n$ together with a closed 2-form ω such that $\omega \wedge \dots \wedge \omega$ (n factors) is nowhere zero. Each phase space T^*C is a symplectic manifold. Similarly, a contact manifold (M, θ) is a manifold of dimension $2n + 1$, where n is an integer, with a one-form θ such that the $(2n+1)$ -form $\theta \wedge d\theta \wedge \dots \wedge d\theta$ (n factors $d\theta$) is non-zero everywhere. State space is an example of a contact manifold. (Note: These contact manifolds are called "exact contact manifolds" in Abraham, loc.cit.)

Both a symplectic manifold and a contact manifold have a non-zero form of highest dimension; that is an "element of volume". For example, in euclidean three-space an element of volume is usually written $dx dy dz = dx \wedge dy \wedge dz$ with respect to rectangular coordinates; $r^2 \sin \theta dr d\theta d\phi$ with respect to spherical coordinates, and so on. In general a volume element on an n -dimensional vector space W is a non-zero element $b \in \Lambda_n(W)$. Since the n -th exterior power $\Lambda_n(W)$ is a one-dimensional vector space, any two volume elements b and b' on W are proportional: $b' = rb$, where r is a non-zero number. Now we often speak of "right-handed" and "left-handed" coordinate systems on

