$$f = \begin{cases} gb & \text{on } U \\ 0 & \text{outside of } U \end{cases} .$$

Then f is continuous -- for $V \subset \mathbb{R}$ open, $f^{-1}(V) = (gb)^{-1}(V)$ is open if $0 \notin V$, and if $0 \in V$, $f^{-1}(V) = (gb)^{-1}(V) \cup (M - C')$ is open, since C' is closed by the above remark. Moreover f is smooth, since g and b are both smooth.

The theorem shows that we could have defined a manifold in terms of functions defined on the entire manifold. However, such a definition would make it more difficult to show that certain manifolds (such as tangent bundles) can be constructed by piecing together other manifolds.

Note that the Theorem would also hold for topological manifolds, but does not hold for analytic manifolds, because the bump function cannot be made analytic.

33. Volumes on Symplectic and Contact Manifolds.

Let us now review the standard set-up we use for discussing mechanics on a general differentiable manifold. Configureation space, C, is an n-dimensional manifold whose points correspond, roughly speaking, to "configurations" or "positions" of the mechanical system. Phase space is the cotangent bundle, T^*C, with the canonical 2-form ω; in local coordinates, $\omega = \Sigma\, dp_i \wedge dq^i$. We define event space to be the topological product $C \times I$, where I is an interval of time, that is, an interval on the real line with t as coordinate. A point (c, t) of event space represents the state c at the time t. Finally, state space is defined to be the product manifold $T^*C \times I$, endowed with the canonical one-form given in local coordinates as $\theta = -\Sigma\, p_i \wedge dq^i + dt$.

Now recall from Part I, § 22, that a symplectic manifold (M, ω) is a manifold M of even dimension 2n together with a closed 2-form ω such that $\omega \wedge \ldots \wedge \omega$ (n factors) is nowhere zero. Each phase space T^*C is a symplectic manifold. Similarly, a contact manifold (M, θ) is a manifold of dimension 2n + 1, where n is an integer, with a one-form θ such that the (2n+1)-form $\theta \wedge d\theta \wedge \ldots \wedge d\theta$ (n factors $d\theta$) is non-zero everywhere. State space is an example of a contact manifold. (Note: These contact manifolds are called "exact contact manifolds" in Abraham, loc. cit.)

Both a symplectic manifold and a contact manifold have a non-zero form of highest dimension; that is an "element of volume". For example, in euclidean three-space an element of volume is usually written $dx\,dy\,dz = dx \wedge dy \wedge dz$ with respect to rectangular coordinates; $r^2 \sin\theta\, dr\, d\theta\, d\varphi$ with respect to spherical coordinates, and so on. In general a volume element on an n-dimensional vector space W is a non-zero element $b \in \Lambda_n(W)$. Since the n-th exterior power $\Lambda_n(W)$ is a one-dimensional vector space, any two volume elements b and b' on W are proportional: $b' = rb$, where r is a non-zero number. Now we often speak of "right-handed" and "left-handed" coordinate systems on

Euclidean three-space; similarly, there may be two types of volume elements. To see this, say that b and b' are equivalent if the proportionality constant r is positive. This divides the volume elements up into equivalence classes: $dx\wedge dy\wedge dz = -dx\wedge dz\wedge dy = dz\wedge dx\wedge dy$, so the elements $dx\wedge dy\wedge dz$ and $dz\wedge dx\wedge dy$ are equivalent.

A volume on an n-dimensional manifold M is a form Ω on $\Omega_n(M)$ which is non-zero everywhere on M. Any two volumes Ω and Ω' are related by the formula $\Omega = f\Omega'$, where f is a smooth non-zero real-valued function on M. If f is positive everywhere, call Ω and Ω' equivalent. Then an orientation of M is defined to be an equivalence class of volumes. Since M may not have a volume in the first place, M may not be orientable; however, we have seen that symplectic manifolds and exact contact manifolds are orientable. A mőbius strip is an example of a non-orientable manifold.

Let Ω be a volume on the n-dimensional manifold M. If X is a vector field on M, the Lie derivative $L_X\Omega$ is another n-form. But any two n-forms at a point are proportional. Thus there is a smooth function f such that $L_X(\Omega) = f\Omega$. We write $f = \text{div } X$; notice that div X depends on the choice of a volume element. Does this agree with the usual notion of the divergence of a vector field? In the situation $M = \mathbb{R}^n$, with coordinates $x^1, \ldots, x^n$, we can write $\Omega = dx^1\wedge \ldots \wedge dx^n$, and $X = \Sigma X^i \dfrac{\partial}{\partial x^i}$. Then

$$L_X\Omega = \sum_i dx^1 \wedge \dots \wedge L_X dx^i \wedge \dots \wedge dx^n .$$

But

$$L_X dx^i = \sum_j (X^j \frac{\partial}{\partial x^j})\, dx^i = \frac{\partial X^i}{\partial x^i}\, dx^i .$$

Hence

$$L_X(\Omega) = [\sum_{i=1}^{n} (\frac{\partial X^i}{\partial x^i})]\Omega$$

so $\operatorname{div} X = \Sigma \frac{\partial X^i}{\partial x^i}$, as expected.

Moreover, our generalized definition of divergence proves a suitable extension of the idea of divergence as the infinitesimal change of volume at a point. For (Part I, 24) the derivative $L_X\Omega$ describes the rate of change of the volume along the trajectories of X.

34. Poisson Brackets.

Let (M, ω) be a symplectic manifold, with symplectic coordinates $\{p_i, q^i\}$; if f and g are two real-valued functions on M, the poisson bracket of f and g with respect to the coordinates p_i, q^i is the smooth function defined by

$$\{f, g\} = \sum_{i=1}^{n} (\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}) .$$

It can be shown that the value of the poisson bracket $\{f, g\}$ of f and g does not depend on the choice of coordinates; however, we seek an invariant description of this function, since it will help us find a formulation of the laws of mechanics leading naturally to quantum mechanics.

We now develop the general algebraic machinery needed for this invariant definition of the Poisson brackets. We previously defined the exterior derivative d, which takes k-forms into (k+1)-forms for every non-negative integer k. Given a vector field X, there is likewise an operation i_X mapping (k+1)-forms ω to k-forms $i_X\omega$:

$$\begin{aligned} i_X\omega(X_1,\dots,X_k) &= (k+1)\,\omega(X,X_1,\dots,X_k) \\ &= \sum_i (-1)^i\,\omega(X_1,\dots,X_{i-1},X,X_i,\dots,X_k). \end{aligned}$$

Finally, the Lie derivative L_X takes k-forms to k-forms. We can now state three identities:

(1) $i_X d + d\,i_X = L_X$ ("homotopy identity");

(2) $L_X(\eta(X_1,\dots,X_k)) = (L_X\eta)(X_1,\dots,X_k) + \sum_{i=1}^{n} \eta(X_1,\dots,L_X X_i,\dots,X_k)$,

where η is a k-form, so that $\eta(X_1,\dots,X_k)$ is a function on M;

(3) $2\,d\omega(X,Y) = L_X\omega(Y) - L_Y\omega(X) - \omega([X,Y])$,

where ω is a one-form.

Proof. For (1), we first notice that i_X is an antiderivation: that is if α is a k-form,

$$i_X(\alpha\wedge\beta) = (i_X\alpha)\beta + (-1)^k\alpha\wedge(i_X\beta).$$

This is an easy computation from the definition of i_X.

Now we prove $i_X d\alpha + d i_X\alpha = L_X\alpha$ by induction on k: for a function f (a 0-form), $i_X f$ is defined to be zero, and we have $i_X df = \langle df, X\rangle = L_X f$. Assuming the result true for k-forms, write a general (k+1)-form, α, as $\Sigma\, df_i \wedge \omega_i$; by linearity it will suffice to prove the result for each summand.

But

$$L_X(df\wedge\omega) = (L_X df)\wedge\omega + df\wedge(L_X\omega),$$

while

$$\begin{aligned} & i_X d(df\wedge\omega) + di_X(df\wedge\omega) \\ &\quad = -i_X(df\wedge d\omega) + d(i_X df\wedge\omega - df\wedge i_X\omega) \\ &\quad = -(i_X df)\wedge d\omega + df\wedge(i_X d\omega) \\ &\qquad\qquad + (di_X df)\wedge\omega + (i_X df)\wedge d\omega + df\wedge(di_X\omega). \end{aligned}$$

Here the first and fourth terms cancel, giving

$$\begin{aligned} & df\wedge(i_X d\omega) + (di_X df)\wedge\omega + df\wedge(di_X\omega) \\ &\quad = df\wedge L_X\omega + (di_X df\wedge\omega) \quad \text{(by inductive assumption)} \\ &\quad = df\wedge L_X\omega + (d(L_X f)\wedge\omega) = df\wedge L_X\omega + (L_X df)\wedge\omega. \end{aligned}$$

This proves (1).

For part (2), recall that L_X commutes with contractions, while evaluation of η at $(X_1,\ldots,X_k)$ is nothing but the contraction $\delta(\eta\otimes X_1\otimes\ldots\otimes X_k)$. With this observation (2) follows from the fact that L_X is a derivation. To derive (3), we verify the formula for $\omega = g\,dq$, since we can then extend to a general one-form ω by linearity. But since $d\omega = dg\wedge df$, it is easy to show that both sides of (3) reduce to $(L_X g)(L_Y f) - (L_X f)(L_Y g)$. A similar argument shows that for ω any k-form,

$$(4)\quad (k+1)d\omega(X_o,\ldots,X_k) = \sum_{i=0}^{k}(-1)^i L_{X_i}(\omega(X_o,\ldots,\hat{X}_i,\ldots,X_k)) + \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_o,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k).$$

(Here the $\wedge$ over X_i means "omit X_i".)

Now let (M, ω) be a symplectic manifold. ω induces linear mappings (§21) $X \rightarrow X^\flat$ and $\alpha \longrightarrow \alpha^\#$ taking vector fields into co-vector fields (= 1-forms), and vice versa. We may now define the poisson bracket of two one-forms, α and β, by

<u>Definition.</u> $\{\alpha, \beta\} = -[\alpha^\#, \beta^\#]^\flat$.

In other words, we turn the forms temporarily into vector fields, take the Lie bracket, and return to the space of forms. The minus sign is chosen for convenience in proving such formulas as

<u>Proposition.</u> $\{\alpha, \beta\} = -L_{\alpha^\#}\beta + L_{\beta^\#}\alpha + d(i_{\alpha^\#} L_{\beta^\#}\omega)$.

<u>Proof.</u> ω is closed, hence by (4) above

$$0 = 3d\omega(X, Y, Z) = L_X(\omega(Y, Z)) + L_Y(\omega(Z, X)) + L_Z(\omega(X, Y))$$
$$- \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X).$$

If we let $X = \alpha^\#$, $Y = \beta^\#$, and recall that, by the definition of $\#$, we have $\omega(\alpha^\#, Y) = \frac{1}{2}\alpha(Y)$, the above equation becomes

$$0 = L_{\alpha^\#}(\tfrac{1}{2}\beta(Z)) - L_{\beta^\#}(\tfrac{1}{2}\alpha(Z)) + L_Z(\omega(\alpha^\#, \beta^\#))$$
$$+ \omega(\{\alpha, \beta\}^\#, Z) + \omega([\alpha^\#, Z], \beta^\#) - \omega([\beta^\#, Z], \alpha^\#).$$

Therefore

$$-L_{\alpha^\#}(\tfrac{1}{2}\beta(Z)) + L_{\beta^\#}(\tfrac{1}{2}\alpha(Z)) - L_Z(\omega(\alpha^\#, \beta^\#))$$
$$= \tfrac{1}{2}(\{\alpha, \beta\})Z - \tfrac{1}{2}\beta[\alpha^\#, Z] + \tfrac{1}{2}\alpha[\beta^\#, Z]$$
$$= \tfrac{1}{2}(\{\alpha, \beta\})Z - \tfrac{1}{2}\beta L_{\alpha^\#}Z + \tfrac{1}{2}\alpha L_{\beta^\#}Z .$$

Now the proposition follows from the three identities

$$L_{\alpha^{\#}}(\tfrac{1}{2}\beta(Z)) = (\tfrac{1}{2}L_{\alpha^{\#}}\beta)Z + \tfrac{1}{2}\beta(L_{\alpha^{\#}}Z)$$

$$-L_{\beta^{\#}}(\tfrac{1}{2}\alpha(Z)) = -\tfrac{1}{2}(L_{\beta^{\#}}\alpha)Z - \tfrac{1}{2}\alpha(L_{\beta^{\#}}Z)$$

$$-2L_Z(\omega[\alpha^{\#},\beta^{\#}]) = d(i_{\alpha^{\#}}i_{\beta^{\#}}\omega)Z,$$

of which the first two are merely (2), above, and the third is a consequence of the equation $i_X\omega(Y) = 2\omega(X, Y)$.

Corollary 1. If β is closed then $\{\alpha, \beta\} = L_{\beta^{\#}}\alpha$.

Proof. By the homotopy identity,

$$\begin{aligned} L_{\alpha^{\#}}\beta &= i_{\alpha^{\#}}d\beta + di_{\alpha^{\#}}\beta \\ &= 0 + 2d(\tfrac{1}{2}\beta(\alpha^{\#})) = 2d(\omega(\beta^{\#}, \alpha^{\#})) = d(i_{\alpha^{\#}}i_{\beta^{\#}}\omega) \end{aligned}$$

Now use the proposition.

Corollary 2. If α and β are closed, $\{\alpha, \beta\} = L_{\beta^{\#}}\alpha = -L_{\alpha^{\#}}\beta = 2d(\omega(\beta^{\#}, \alpha^{\#}))$.

Corollary 3. If α and β are closed, $\{\alpha, \beta\}$ is exact.

For $\{\alpha, \beta\} = d(2\omega(\beta^{\#}, \alpha^{\#}))$.

Now by using #, we see that each function f on M determines a vector field $X_f = (df)^{\#}$.

Corollary 4. If f and g are smooth functions on M, then

$$\begin{aligned} \{df, dg\} &= L_{X_g}(df) = d(L_{X_g}f) \\ &= -L_{X_f}(dg) = -d(L_{X_f}g) \\ &= 2d(\omega(X_g, X_f)). \end{aligned}$$

Definition. The Poisson bracket of the functions f and g is

$$\{f, g\} = L_{X_g} f \qquad (X_g = (dg)^{\#}). \text{ (Hence } d\{f, g\} = \{df, dg\}).$$

Proposition. $L_{X_g} f = -L_{X_f} g = -2\omega(X_g, X_f)$.

Proof.
$$\begin{aligned} L_{X_g} f &= \langle df, X_g \rangle = \langle df, dg^{\#} \rangle = 2\omega(df^{\#}, dg^{\#}) \\ &= 2\omega(X_f, X_g) = -2\omega(X_g, X_f) = -2\omega(dg^{\#}, df^{\#}) \\ &= -\langle dg, df^{\#} \rangle \\ &= -L_{X_f} g. \end{aligned}$$

In particular, $L_{X_g} f = 0$ if and only if $L_{X_f} g = 0$. Thus f is constant on the trajectories of g if and only if g is constant on the trajectories of f. (By the trajectories of f we mean those of the vector field X_f.)

Of course, we must check that this definition agrees with our coordinate-wise notion of poisson bracket. Let the symplectic coordinates be $\{p_i, q^i\}$. This means that $\omega = \Sigma\, dp_i \wedge dq^i$. Any 1-form α can be written

$$\alpha = \Sigma\, h_i dq^i + \Sigma\, k^j dp_j ,$$

while any vector field X can be written

$$X = \Sigma\, X^i \frac{\partial}{\partial q^i} + \Sigma\, T^j \frac{\partial}{\partial p_j} .$$

Then we have seen that

$$\begin{aligned} X^{\flat} &= \Sigma\, T^i dq^i - \Sigma\, X^i dp_i \\ \alpha^{\#} &= -\Sigma\, k^i \frac{\partial}{\partial q^i} + \Sigma\, h_i \frac{\partial}{\partial p_i} . \end{aligned}$$

Thus $\{f, g\} = -L_{X_f} g = \Sigma\left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial p_j} \right)$ as expected.

Since our poisson bracket $\{\ ,\ \}$ was defined invariantly from the 2-form ω, the formula holds for any symplectic (= canonical) coordinates. In particular, this formula gives the poisson bracket of any two coordinate functions. We can deduce hence that a set of $2n$ functions $Q^1, \ldots, Q^n, P_1, \ldots, P_n$ on a symplectic manifold are symplectic coordinates if and only if they satisfy the relations

$$\{P_i, Q_j\} = \delta_{ij}$$

$$\{P_i, P_j\} = \{Q_i, Q_j\} = 0$$

for all i and j.

One can also prove that a transformation $\varphi : (M, \omega) \longrightarrow (M, \omega)$ is symplectic if and only if it preserves all poisson brackets of functions.

Proposition. For any three smooth functions on a symplectic manifold

$$\{f, \{g, h\}\} + \{g\{h, f\}\} + \{h, \{f, g\}\} = 0.$$

This asserts that the set of smooth functions is a Lie algebra under the poisson bracket $\{\ ,\ \}$.

Proof.

$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$

$$= L_{X_f} L_{X_g} h - L_{X_g} L_{X_f} h + L_{X_{\{f, g\}}} h$$

and this is zero, because

$$X_{\{f, g\}} = (d\{f, g\})^{\#} = \{df, dg\}^{\#} = -\{df^{\#}, dg^{\#}\} = -[X_f, X_g].$$

Here is an application of the antisymmetry of the poisson bracket. Consider k particles moving in three dimensional space. Their position is then specified by 3k coordinates

$$(x_1, y_1, z_1; x_2, y_2, z_2; \dots, x_k, y_k, z_k),$$

so that the configuration space C is $\mathbb{R}^{3k}$. In the corresponding phase space $M = T^*C$ we can write down the Hamiltonian function H in terms of the potential energy V and the usual kinetic energy of the 3k particles. If we assume that V depends only on the distances between particles, then the Hamiltonian H is left fixed by the transformations of M induced by rigid motions, like translations and rotations, in $\mathbb{R}^3$. Let X_g be the vector field corresponding to such a translation; then X_g leaves H invariant. By anti-symmetry, X_H must leave g invariant; that is, since the system moves along the trajectories of H, g is a constant of the motion. For translations, g turns out to be the linear momentum, while for rotations g is angular momentum. We have just derived the familiar conservation-of-momentum laws. In general, any function f with $\{f, H\} = 0$ is a constant of the motion.

35. Submanifolds and Immersions.

We will study "energy surfaces" (submanifolds of constant energy); for this we need some facts about submanifolds. In a number of places in these lectures we have used (and will be using) the theorem below and its corollaries. (Here $Df(m)$ is the map induced by f on the tangent space at the point m.)

Theorem. (Inverse Function Theorem): Let $M \xrightarrow{f} N$ be a smooth function. If $Df(m)$ is an isomorphism, then f is a local diffeomorphism at m; i.e., there are neighborhoods U of m and V of fm such that $f(U) = V$ and $f|U: U \longrightarrow V$ has a smooth inverse.

Corollary 1. (Implicit Function Theorem): Let $M \xrightarrow{f} N$ be a smooth function. If $Df(m)$ is a surjection, then f is locally a projection; i.e., there are charts (U, ϕ) at m and (V, ψ) at fm such that

$\phi U = U' \times V'$

$\psi V = V'$

and $\psi \circ f \circ \phi^{-1}$ is the projection of $U' \times V'$ onto V'.

$$\begin{tikzcd}
M \arrow[r, "f"] & N \\
\cup & \cup \\
m \in U \arrow[d, "\cong"', "\phi"] & fm \in V \arrow[d, "\cong"', "\psi"] \\
U' \times V' \arrow[r, "\psi \circ f \circ \phi^{-1}"] & V'
\end{tikzcd}$$

Corollary 2. Let $M \xrightarrow{f} N$ be a smooth function. If $Df(m)$ is an injection, then f is locally an injection; i.e., there are charts (U, ϕ) at m and (V, ψ) at fm such that $\phi U = U'$, $\psi V = U' \times V'$, and $\psi \circ f \circ \phi^{-1}$ is the injection of U' into $U' \times V'$ as $U' \times 0$.

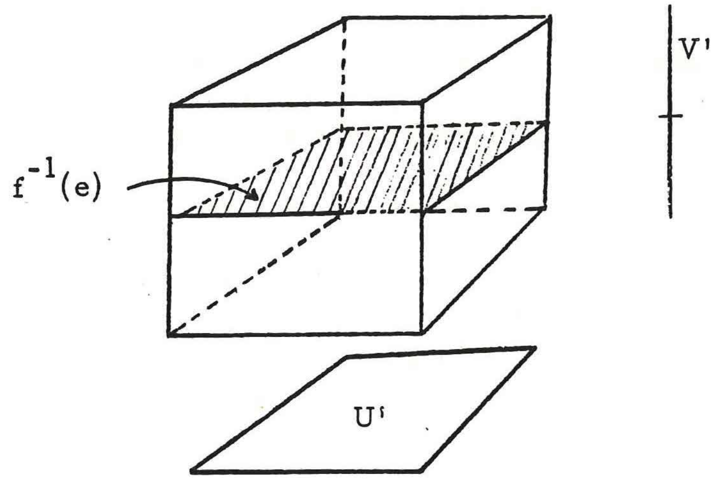
In what follows, M and N are manifolds and $a \in M$. An <u>immersion</u> of M in N is a smooth function $M \xrightarrow{f} N$ with the property: for each $f(a) \in N$, there is a chart at $f(a)$ with coordinates $q^1, \ldots, q^n$ such that $q^1 f, \ldots, q^d f$ are coordinates for a chart at a, for some $d \leq n$.

By Corollary 2 above, a smooth function $M \xrightarrow{f} N$ is an immersion if and only if $Df(a)$ is an injection for every $a \in M$. An <u>embedding</u> is an immersion which is a homeomorphism onto its image endowed with the subspace topology. A weaker notion of embedding which sometimes is used is an immersion that is an injection (1-1 function); but the stronger sense seems to be what we want for mechanics. If $M \subset N$ and the inclusion is an embedding, then M is a <u>submanifold</u> of N. One last definition: the point $e \in N$ is a <u>regular value</u> of f if and only if the Jacobian $Df(a)$ has maximum rank for every a such that $f(a) = e$. Since $Df(a)$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ it will have maximum rank when it is surjective if $m \geq n$ and when it is injective if $m \leq n$.

<u>Theorem</u>. Let N, P be manifolds and $N \xrightarrow{f} P$ a smooth function. If $e \in P$ is a regular value of f, then $f^{-1}(e)$ is a submanifold of N.

<u>Proof</u>. Since the inclusion $f^{-1}(e) \subset$ ~~M~~ N is clearly a homeomorphism onto its image we need only show the immersion property. Take $a \in N$ so that $f(a) = e$.

Case (i). $Df(a)$ is surjective: Then Corollary 1 of the Inverse Function Theorem gives a chart (U, ϕ) at a such that $\phi(f^{-1}(e) \cap U) = U'$ and $\phi U = U' \times V'$. Locally the picture shows what the manifold M looks like near a.



Case (ii). $Df(a)$ is injective: Then f is an injection near a, so $f^{-1}(e)$ is a set of isolated points each of which is a submanifold. Thus the union of these is a submanifold, so the theorem is proved.

If $P = \mathbb{R}$ in the theorem, it is useful to have explicit local coordinates for a point a such that $f(a) = e$. Let $q^1, \ldots, q^n$ be any coordinates around a; then $Df(a) = \left(\frac{\partial f}{\partial q^1}, \ldots, \frac{\partial f}{\partial q^n}\right)_a$ where e is a regular value means one of the entries is non-zero, say $\frac{\partial f}{\partial q^n}$. The change of coordinates $\begin{matrix} q^i \longrightarrow q^i & 1 \leq i \leq n-1 \\ q^n \longrightarrow f & \end{matrix}$ has Jacobian matrix

$$\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & \frac{\partial f}{\partial q^n} \end{pmatrix}$$

so $q^1, \ldots, q^{n-1}, f$ are coordinates around a in N such that $q^1, \ldots, q^{n-1}$ are coordinates around a in $f^{-1}(e)$.

36. Invariants on a Symplectic Manifold.

We first study quantities invariant under a vector field on any manifold.

Definition. Let M be a manifold and X a vector field on M; then a k-form α is invariant under X if and only if $L_X\alpha = 0$.

We have the equivalences

$$L_X\alpha = 0 \text{ iff } F_t^*\alpha = \alpha, \text{ where } F \text{ is a flow of } X,$$
$$\text{iff } \alpha \text{ is constant on integral curves of } X.$$

The following properties are easy to prove.

(i) α invariant under X implies $i_X\alpha$ and $d\alpha$ are invariant under X.

(ii) α and β invariant under X implies $\alpha\wedge\beta$ is invariant under X.

(iii) α invariant under X and $L_XY = 0$ implies $i_Y\alpha$ is invariant under X,

where X, Y are vector fields and α, β forms.

Proof of (i) for $i_X\alpha$: By the "homotopy identity" (1) of 33, $L_X = i_Xd + di_X$, thus

$$L_X(i_X\alpha) = i_X d i_X\alpha + d i_X i_X\alpha = -i_X i_X d\alpha + d i_X i_X\alpha.$$