

$X(a) = 0$ , the Jacobian

$$\left\| \frac{\partial x^i}{\partial q^j} \right\|$$

provides a linear approximation to  $X$  near  $a$ . For example, in dimension 1, a Taylor expansion gives

$$\frac{dq}{dt} = \lambda q + \text{higher order terms,}$$

where  $\lambda$  is the eigenvalue of the Jacobian. Thus

$$\frac{dq}{dt} = \lambda q$$

gives a first approximation to solutions near  $a$ . In higher dimensions it is generally possible to choose coordinates  $q^1, \dots, q^n$  so that

$$\frac{dq^i}{dt} = \lambda^i q^i$$

give first approximations to the solutions near  $a$ , where the  $\lambda^i$  are the eigenvalues of the Jacobian matrix near  $a$ .

2) Closed orbits. Nearby trajectories may be studied by taking a normal cross-section to the given closed orbit. Again, suitable eigenvalues determine the behavior; they are obtained by mapping the cross-section on itself by following along trajectories going "once around" the orbit.

### Structural Stability -- René Thom \*

The purpose of mechanics is to describe the motion of physical bodies. Recently the theories developed for this aim have also been used to study chemical and even biological phenomena.

Two separate theories of mechanics have evolved. Time reversible mechanics is based on the assumption that the time parameter can be reversed without changing the qualitative aspects of the phenomenon being studied. Vibration without damping is an example of such a phenomenon. Time reversible mechanics has been dominated by Hamiltonian theory and is centered on the concept of Invariance of Energy. Time reversible mechanics suffers from the defect that it is in most cases an idealization of nature. Time-irreversible mechanics is more true to nature but has been studied less than time-reversible mechanics. It is dominated by the study of gradient-like systems and centered on the concept of Increase of Entropy. More explicitly, if  $X$  is a vector field on a phase space  $M$ , then Increase of Entropy is satisfied if there exists a function  $S: M \rightarrow \mathbb{R}$  (the entropy function) such that  $S(m_t)$  is monotone increasing, where  $\frac{dm_t}{dt} = X$ . Otherwise put,  $X$  is transversal in an increasing direction to the level varieties of  $S$ .

---

(\*) Prof. Thom wishes to thank Prof. S. MacLane for having been given this opportunity to expose some favorite ideas in the field of Geometrical Mechanics.

39. Gradient Vector Fields.

Let  $M$  be a manifold with a Riemannian metric  $\langle, \rangle$ . Then there is a correspondence between vector fields  $X$  on  $M$  and 1-forms  $\alpha_X$  on  $M$  given by

$$\langle X, u \rangle = \alpha_X(u) \text{ for } u \in T_m M,$$

at each  $m \in M$ . (This is just the correspondence induced by the isomorphism of tangent and cotangent bundles given by  $\langle, \rangle$ .) If  $\alpha_X = dU$ , then we set

$$X = \text{grad } U.$$

The situation may be described by saying that  $X$  is orthogonal to the level surfaces of  $U$ . This is just like an entropy situation, particularly since we are free to choose a convenient Riemann metric.



At regular points (that is, points where  $X \neq 0$ ) there is little difficulty in determining the nature of the trajectories. At a singular point the situation becomes more complicated. The behavior of trajectories near a singular point of a gradient field might be as in the picture below.



