$X(a) = 0$, the Jacobian

$$\left\| \frac{\partial x^i}{\partial q^j} \right\|$$

provides a linear approximation to X near a. For example, in dimension 1, a Taylor expansion gives

$$\frac{dq}{dt} = \lambda q + \text{higher order terms},$$

where λ is the eigenvalue of the Jacobian. Thus

$$\frac{dq}{dt} = \lambda q$$

gives a first approximation to solutions near a. In higher dimensions it is generally possible to choose coordinates $q^1, \ldots, q^n$ so that

$$\frac{dq^i}{dt} = \lambda^i q^i$$

give first approximations to the solutions near a, where the λ^i are the eigenvalues of the Jacobian matrix near a.

2) Closed orbits. Nearby trajectories may be studied by taking a normal cross-section to the given closed orbit. Again, suitable eigenvalues determine the behavior; they are obtained by mapping the cross-section on itself by following along trajectories going "once around" the orbit.

Structural Stability -- René Thom*

The purpose of mechanics is to describe the motion of physical bodies. Recently the theories developed for this aim have also been used to study chemical and even biological phenomena.

Two separate theories of mechanics have evolved. Time reversible mechanics is based on the assumption that the time parameter can be reversed without changing the qualitative aspects of the phenomenon being studied. Vibration without damping is an example of such a phenomenon. Time reversible mechanics has been dominated by Hamiltonian theory and is centered on the concept of Invariance of Energy. Time reversible mechanics suffers from the defect that it is in most cases an idealization of nature. Time-irreversible mechanics is more true to nature but has been studied less than time-reversible mechanics. It is dominated by the study of gradient-like systems and centered on the concept of Increase of Entropy. More explicitly, if X is a vector field on a phase space M, then Increase of Entropy is satisfied if there exists a function $S: M \longrightarrow \mathbb{R}$ (the entropy function) such that $S(m_t)$ is monotone increasing, where $\frac{dm_t}{dt} = X$. Otherwise put, X is transversal in an increasing direction to the level varieties of S.

(*) Prof. Thom wishes to thank Prof. S. MacLane for having been given this opportunity to expose some favorite ideas in the field of Goemetrical Mechanics.

39. Gradient Vector Fields.

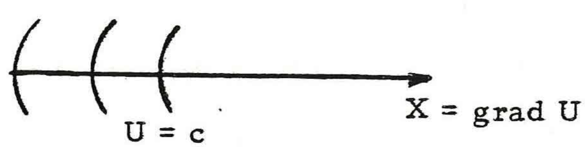
Let M be a manifold with a Riemannian metric $<,>$. Then there is a correspondence between vector fields X on M and 1-forms α_X on M given by

$$<X,u> = \alpha_X(u) \quad \text{for } u \in T_mM,$$

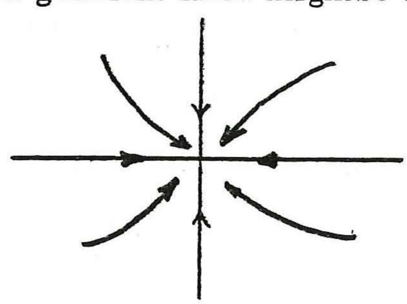
at each $m \in M$. (This is just the correspondence induced by the isomorphism of tangent and cotangent bundles given by $<,>$.) If $\alpha_X = dU$, then we set

$$X = \text{grad } U.$$

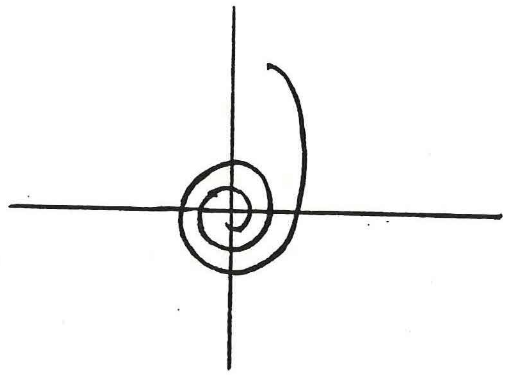
The situation may be described by saying that X is orthogonal to the level surfaces of U. This is just like an entropy situation, particularly since we are free to choose a convenient Riemann metric.



At regular points (that is, points where $X \neq 0$) there is little difficulty in determining the nature of the trajectories. At a singular point the situation becomes more complicated. The behavior of trajectories near a singular point of a gradient field mightbe as in the picture below.



In a more general situation a focus, as in the diagram below, might occur; this cannot happen with a gradient field, however. (This is because of the symmetry of the second order derivatives $\dfrac{\partial^2 V}{\partial x_i \partial x_j}$



40. Qualitative Dynamics

The classical approach to dynamics was to try to solve the Hamilton-Jacobi equation

$$\frac{dm}{dt} = X(m)$$

explicitly. This appraoch was beset with difficulties. Frequently, X is not known exactly -- say, if not all the forces acting on a system are explicitly known. In this case, empirical formulae are used to approximate the desired information. Given a known vector field X, there are not always adequate means of integration available, as for the three-body problem of Newtonian mechanics. Then the solution must be approximated. To approximate reasonably, it is necessary to know how much a slight perturbation of X affects the global solution.

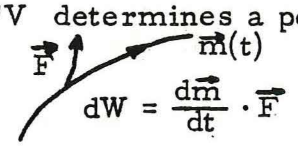
In the 1880's, Poincare introduced the study of Qualitative Dynamics, which aims to describe solutions rather than find them explicitly. Once the geometric picture formed by the trajectories is found, one can pose the structural stability problem: to determine whether this picture is invariant up to homeomorphism under small perturbations of X. We whall examine these problems for the case of gradient fields. We will deal with a potential function

$$M \xrightarrow{V} \mathbb{R}$$

and the corresponding vector field

$$X = -\operatorname{grad} V.$$

First, a digression concerning the nature of potential functions. If the vector field X corresponds as above to the one-form α_X, then $\alpha_X(z)$ is called the work of the field of force z. If α is closed -- $d\alpha = 0$ -- then we say "V determines a potential." Either of two cases may occur:



1) If α_X is homologous to zero in the one dimensional cohomology group then $H^1(M, \mathbb{R})$, then $\alpha = d(-V)$.

2) If α_X is not homologous to zero in $H^1(M, \mathbb{R})$, then the potential is "multi-valued"; i.e., it is defined up to multiples of periods. We shall consider only the first case.

Definition. $p \in M$ is a critical point of X if $X(p) = 0$; i.e., if p is a singular point of X.

If $X = -\operatorname{grad} V$, p will be a singular point of X if and only if $dV_p = 0$; i.e., if and only if p is a critical point of V in the usual sense.

Suppose M is compact. Let

$$h_t: M \xrightarrow{\text{onto}} M$$

be a one-parameter family of diffeomorphisms obtained by integrating X, such that $h_t(m_o) = m_t$. (That is, h_t is a flowbox for X.) Consider the h_t's applied to one point $m \in M$; since M is compact, the resulting set $\{h_t(m)\}$ has limit points.

Claim: Any limit point of a trajectory m_t for X is a critical point of V.

Proof. Suppose q is a limit point of m_t. Then the trajectory m_t keeps coming back near q. By definition, $V(m_t)$ is decreasing. Since M is compact, $\{V(m_t)\}$ has a lower bound. Thus we must have $V(m_t) \downarrow V(q)$ as $t \to \infty$. If q is not critical, then V has a non-zero gradient at q, so there would be nearby points with values less than $V(q)$, contrary to $V(m_t) \downarrow V(q)$.

Thus we have shown that "all trajectories go to critical sets." The same reasoning shows that "all trajectories start from critical sets."

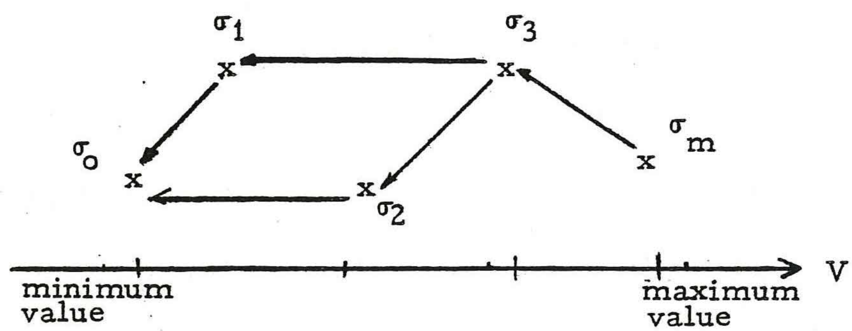
Definition: The critical set of X is $(dV)^{-1}(0)$. Thus, if V is C^1, then the critical set is closed.

Is V constant on any connected component of the critical set? If the components are all differentiable-arc-connected (i.e., if any two points in the same component can be connected by a differentiable arc lying in that component), then the answer is "yes".

<u>Theorem</u> (A.P. Morse)[1]: If V is of class C^m where $m \geq n = \dim M$, then V is constant on any component of the critical set.

There is a counterexample, due to Whitney, showing that V can be non-constant on a component if its class is too low.

When V is constant on each component of the critical set, the components can be ordered via the values which V assumes on them. The structure of the gradient field may then be described by the following procedure. Let σ_i be the components of the critical set. To each σ_i associate a point above the value in $\mathbb{R}$ which V assumes on σ_i. Draw an arrow from σ_i to σ_j whenever there is a trajectory for X in M which starts in a neighborhood of σ_i and ends in a neighborhood of σ_j. Such a graph might look like the following:



In what follows, assume V is C^∞. Let $V(x_i)$ denote the potential as a function of the coordinates x_i. Suppose the origin is a critical point -- i.e.,

$$\frac{\partial V}{\partial x_1}(0) = \dots = \frac{\partial V}{\partial x_n}(0) = 0.$$

---

1) See for instance G. de Rham, Varietes Differentiables, Hermann Paris, Th. 9, p. 10; or S. Sternberg, Lectures on Differential Geometry, Prentice Hall -- Sard's Theorem --Theorem 3.1, p. 47.

The origin is a <u>non-degenerate</u> critical point if the Hessian $\left\| \frac{\partial^2 V}{\partial x_i \partial x_j} \right\|$ is non-zero there. Equivalently, if

$$V(x_i) = V(0) + \varphi_2(x_i) + \ldots$$

is the Taylor expansion for V, then 0 is a non-degenerate critical point if and only if the quadratic form $\varphi_2(x_i)$ is non-degenerate. In this case, φ_2 can be reduced to the sum of squares of linearly independent forms:

$$\varphi_2(x_i) = \sum \pm x_i^2 = -x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2 .$$

Then k is the <u>index</u> of the quadratic form and, along with the fact that 0 is a non-degenerate critical point, it is invariant under changes of coordinates.

Suppose $V(x_i)$ is a potential function which admits 0 as a non-degenerate critical point. Perturb V slightly to

$$V(x_i) + \delta V(x_i),$$

where $\delta V(x_i)$ is small in a suitable norm. Then the new function has near 0 a non-degenerate critical point of the same index.

To show this we introduce the "auxiliary map"

$$\mathbb{R}^n \xrightarrow{G} \mathbb{R}^n$$

given by $x_i \rightsquigarrow u_i = \frac{\partial V}{\partial x_i}(x_1, \ldots, x_n)$. Then the critical set is just $G^{-1}(0)$, and 0 is non-degenerate if and only if G has rank n at 0. Perturb V slightly in the C_2 norm; this perturbs G to a new map G' which is a C_1 approximation of G. Therefore G' has rank n in some neighborhood of 0, so $(G')^{-1}(0)$ must contain a point in this neighborhood. The invariance of index follows from the continuity of the second derivatives.

Thus we have that "non-degenerate critical points are invariant under small perturbations". This theorem is due to M. Morse. We next state without proof the following result of M. Morse:

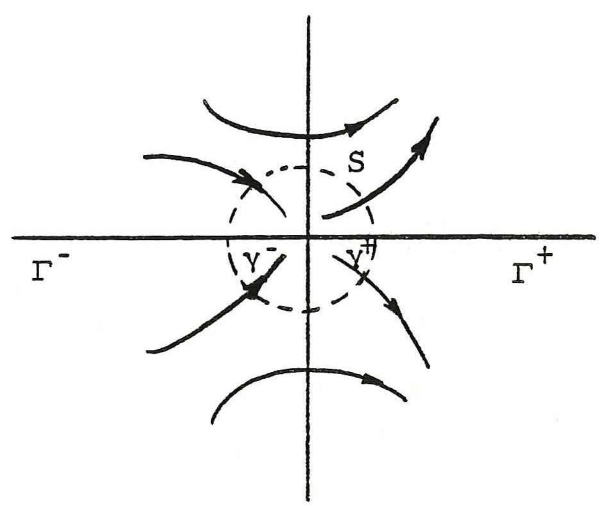
Lemma. (M. Morse): If $V(x_i) = V(0) + \varphi_2(x_i) + \varphi_3(x_i) + \ldots$, then there is a change of coordinates $x_i \to x_i'$ near 0 such that

$$V(x_i') = V(0) + \sum \pm x_i'^2 ,$$

i.e., V can always be reduced to a quadratic form (plus a constant).

If a critical point p is non-degenerate, we can get a good description of the gradient field near p. Before attempting this, we state a related conjecture:

Suppose f is an analytic real-valued function on $\mathbb{R}^n$ with 0 as an isolated singular point; let (f_{x_i}) be the ideal (in the ring of analytic functions at 0) generated by the first partial derivatives of ~~R~~. Then (f_{x_i}) contains a power of the ideal generated by the coordinate functions; in other words, any monomial of sufficiently high degree is a sum of multiples of the first partials of f. Consider the set of trajectories of grad f. There will generally be a set Γ^- of trajectories tending toward the origin, and another, Γ^+, of trajectories emanating from the origin. The problem is to show that Γ^- and Γ^+ have a nice topological structure; in particular, that they may be triangulated, preferably by a triangulation which can be extended to the whole space. It has so far been shown that if Γ^+ and Γ^- are cut by a suffi-



ciently small sphere S, yielding sets γ^+ and γ^-, then γ^+ is a deformation retract of $S-\gamma^-$, and vice-versa.

Now suppose 0 is a non-degenerate critical point of $F(x_i)$

$$F_{x_i}(0) = 0$$

and

$$F_{x_i x_j}(0) \neq 0.$$

What can be said about the qualitiative structure of the gradient field near 0?

By M. Morse's theorem, we can choose coordinates $x_1, \ldots, x_k$, $y_1, \ldots, y_{n-k}$ around the critical point 0 so that

$$F = \sum_{i=1}^{k} x_i^2 - \sum_{j=1}^{n-k} y_j^2 .$$

<u>Special case:</u> Take as the Riemann metric about 0

$$ds^2 = dx^2 + dy^2 = \sum dx_i^2 + \sum dy_j^2 .$$

Now

$$\frac{1}{2} F_{x_i} = x_i ,$$

$$\frac{1}{2} F_{y_j} = -y_j ,$$

and

$$\frac{1}{2} \operatorname{grad} F = \sum F_{x_i} \frac{\partial}{\partial x_i} - \sum F_{y_j} \frac{\partial}{\partial y_j} ,$$

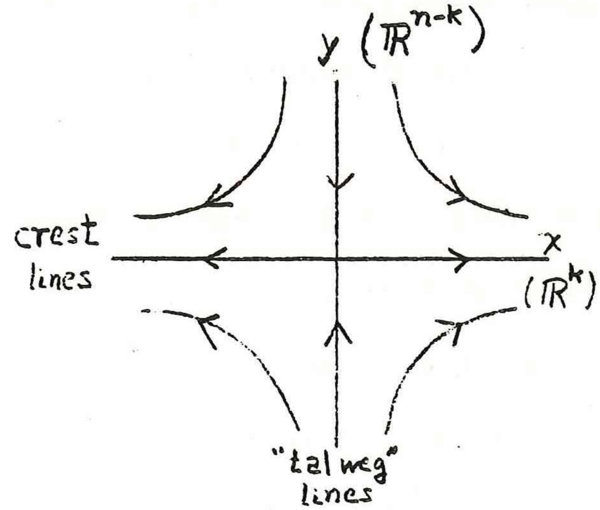
so integrating the system amounts to solving

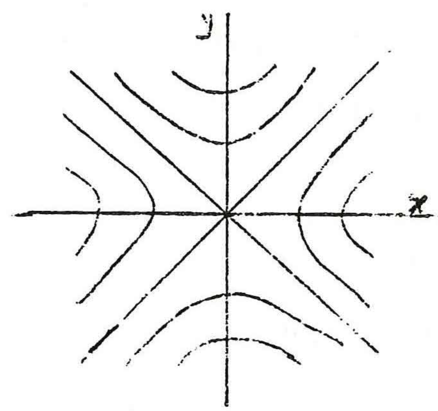
$$\frac{dx_i}{dt} = x_i , \quad \frac{dy_j}{dt} = -y_j$$

which we can readily do to get

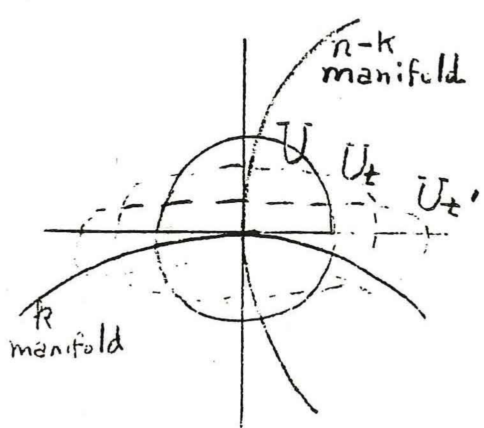
$$x_i = a_i e^{t} , \quad y_j = b_j e^{-t} .$$

The orbits of trajectories tending to or from the origin fill up the "axes" R^{n-k} and $\mathbb{R}^k$, where $x = 0$ and $y = 0$, respectively. The trajectory through any point not in one of these sets does not pass through the origin. In general (if $k \neq 0, n$), there will be a saddle point -- as is pictured in the two-dimensional case for $F = x^2 - y^2$. (The level curves for F itself in the case $n = 2$ are pictured in the middle diagram)



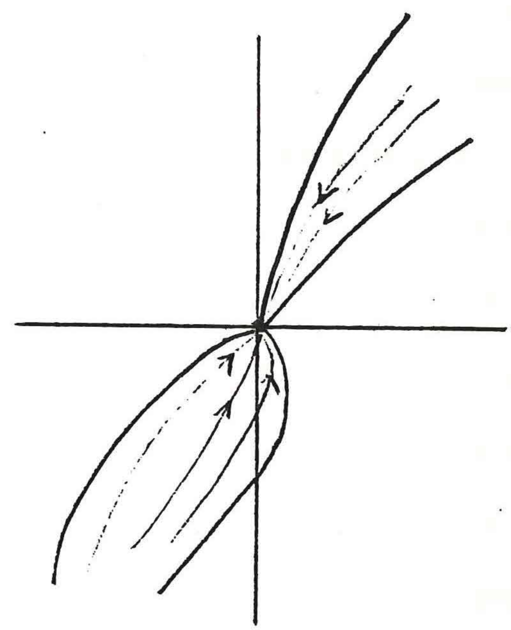


If instead we take an arbitrary Riemann metric in a neighborhood of the critical point, an analysis more delicate than that above will yield the same results. There will be a k-manifold passing through 0 formed by trajectories leaving 0, and an n-k manifold orthogonal to it formed by trajectories tending to 0, where n-k is the index of the critical point. The general method is to take a small neighborhood U of 0 and consider its images $U_t = h_t(U)$,



where the h_t form as above a flow box for X. The point 0 is a fixed point of h_t, so each U_t is a neighborhood of 0. As $t \to \infty$, these neighborhoods stretch along the x-axis, tending toward a limit which is a k-dimensional manifold of trajectories leaving the origin. As $t \to -\infty$, the U_t tend toward the n-k dimensional manifold of trajectories approaching the origin. P. Hartman has done the analysis, that of "unstable critical sets", involved.[3)]

The above results hold only for a C^2 function F. For example, if $F(x, y)$ is a C^1 but not C^2 function of two variables, the trajectories arriving at the origin might form a full sector.



It is known that the homotopy types of the sets of trajectories entering and leaving a critical point are fairly well determined. However, other topological properties may vary widely.

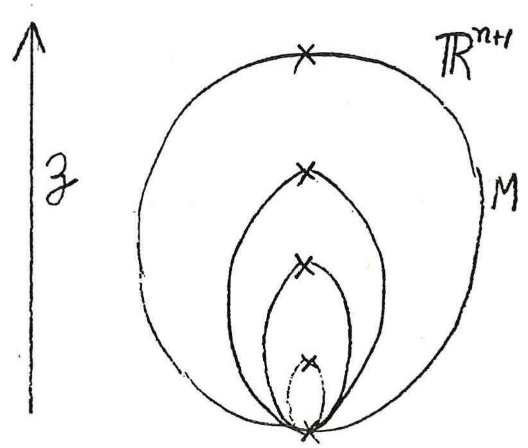
3) P. Hartman - On the local linearization of differential equations, Proceedings Amer. Math. Soc., 1963, 14, pp. 568-73.

41. Morse Theory.

Theorem (M. Morse): Any real-valued function of class C^m on a compact manifold M^n, where $m > n$, can be approximated in the C^n topology by a function which admits only non-degenerate critical points.

Consider $\mathcal{D}^m(M^n)$ = all real-valued C^m functions on M^n, with the C^m topology; it is a Banach space. Morse's theorem says that the set of functions in $\mathcal{D}^m(M^n)$ which admit only non-degenerate critical points is open and dense in $\mathcal{D}^m$.

Now suppose $\mathfrak{z}$ is a function defined on a compact n-manifold M^n imbedded in $\mathbb{R}^{n+1}$ in such a way that $\mathfrak{z}$ is a coordinate in $\mathbb{R}^{n+1}$. By Morse's theorem, we may approximate $\mathfrak{z}$ by a function with only non-degenerate critical points. So without loss of generality, we may assume $\mathfrak{z}$ has only finitely many critical points, each of which is non-degenerate. Associate to each of these the set of "descending trajectories" from that point (i. e., the set of trajectories of -grad $\mathfrak{z}$). These form a k-dimensional cell $\mathfrak{z}_k$, where n-k is the index of the critical point.



The cells $\mathfrak{z}_k$ form a partition of M -- for any point of M lies on a gradient trajectory with $\mathfrak{z}$ increasing, and this trajectory must tend toward a critical point. Thus we have the basic objective of Morse theory a representation of M as a union of k-cells $\mathfrak{z}_k$.

This partition is in fact a C-W complex. (For details, see Milnor's book on Morse theory.)

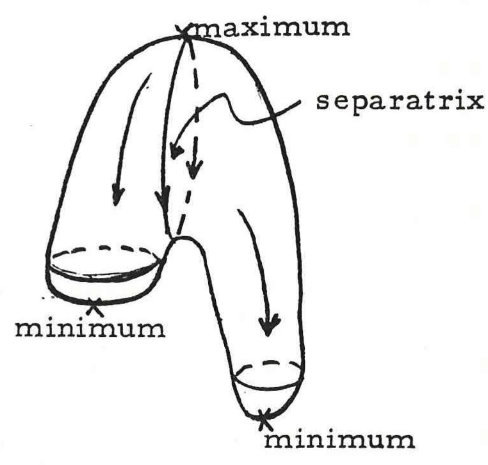
If i_k = the number of critical points of index k, then it can be shown that

$$i_k \geq b_{\cancel{n}}(M^n) \quad , \quad \text{(the Morse inequalities);}$$

k

i.e., i_k is greater than any k-dimensional betti-number of M^n. Consequently, the number of critical points has a lower bound determined by the topological structure of the manifold. This fact has interesting applications in mechanics, since a non-degenerate critical point corresponds to an equilibrium position. However, not all the non-degenerate critical points correspond to stable equilibrium positions -- in fact, it is possible to leave the critical point along an edge of the corresponding cell unless the point is a minimum.

In dimension 3, there may be several minima. Each determines a "basin"; the manifold minus the separatrices between adjacent basins is globally partitioned into basins. In general, we can't expect the separatrices to behave nicely, as will be shown shortly.



Suppose that 0 is a minimum with value 0 and suppose further that this is a quadratic minimum, i.e., if

$$z = \varphi_2(x, y) + \varphi_3(x, y) + \ldots$$

is the Taylor expansion near 0, then there exists an orthogonal change of coordinates such that

$$\varphi_2(x, y) = Ax^2 + By^2,$$

where $A > B > 0$. Consider

$$-\text{grad } z = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y},$$

where $\varphi = \varphi_3 + \ldots$ and

$$X = -Ax - \frac{\partial \varphi}{\partial x}(x, y) \ldots$$

$$Y = -By - \frac{\partial \varphi}{\partial y}(x, y) \ldots .$$

<u>Claim:</u> The trajectories h_t for $-\text{grad } z$ are <u>contracting</u> near 0, i.e., $|h_t| \to 0$ as $t \to \infty$, for trajectories h_t near 0.

<u>Proof.</u> Let $\rho^2 = x^2 + y^2 = |h_t|^2$

$$\frac{d}{dt}(\rho^2) = 2x(-Ax - \frac{\partial \varphi}{\partial x}) + 2y(-By - \frac{\partial \varphi}{\partial y})$$

$$= -2(Ax^2 + By^2) - 2(x \frac{\partial \varphi}{\partial x} + y \frac{\partial \varphi}{\partial y}).$$

Now $|\frac{\partial \varphi}{\partial x}| < M\rho^2$, $|\frac{\partial \varphi}{\partial y}| < M\rho^2$, so the second term in the expression for $\frac{d}{dt}(\rho^2)$ is dominated by ρ^3. But the first term dominates ρ^2. Hence for ρ small,

$$Ax^2 + By^2 > |x \frac{\partial \varphi}{\partial y} + y \frac{\partial \varphi}{\partial y}| ;$$

so

$$\frac{d}{dt}(\rho^2) < 0.$$

An alternative way of looking at the problem would be to consider the inner product between the vector field at a point $h_t = (x, y)$ and the vector from 0 to (x, y). This is just

$$xX + yY = -(Ax^2 + By^2) - (x \frac{\partial \varphi}{\partial x} + y \frac{\partial \varphi}{\partial y}).$$

The above reasoning shows that this inner product is negative for $\rho^2 = x^2 + y^2$ small; hence the angle between the two vectors is between $\pi/2$ and $3\pi/2$, so the vector field always enters any circle of sufficiently small radius ρ .

<u>Claim:</u> All trajectories except the one along the x-axis arrive toward the origin tangential to the y-axis.

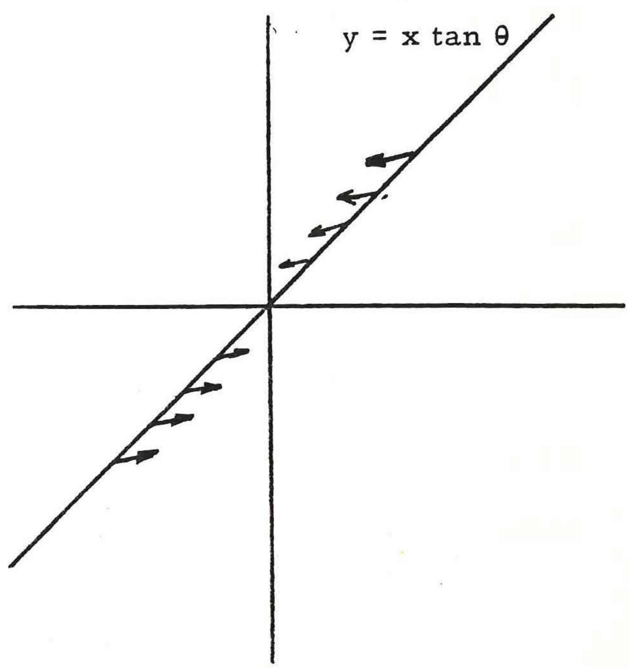
<u>Proof.</u> Consider how the vector field acts along a line $y = x \tan\theta$ through 0:

$$\frac{Y}{X} = \frac{-By - \dfrac{\partial\varphi}{\partial y}}{-Ax - \dfrac{\partial\varphi}{\partial x}}$$

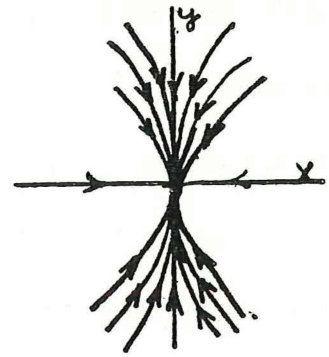
$$= \frac{-B\rho \sin\theta - \dfrac{\partial\varphi}{\partial y}(x, y)}{-A\rho \sin\theta - \dfrac{\partial\varphi}{\partial x}(x, y)}$$

$$= \frac{-B \sin\theta + \varepsilon(\rho)}{-A \cos\theta}$$

for ρ small, where $\varepsilon(\rho) \to 0$ as $\rho \to 0$. For ρ small enough the vector field is pointed at an angle of approximately $\arctan(\frac{B}{A} \tan\theta)$ with the x-axis. Since $B < A$, this angle is smaller than θ. Thus for small ρ, the vector field enters the angle formed by this line and the y-axis. Since B/A is a constant, every trajectory near 0 which is not along the



x-axis must enter every angle containing the y-axis, hence arrives tangential to it.

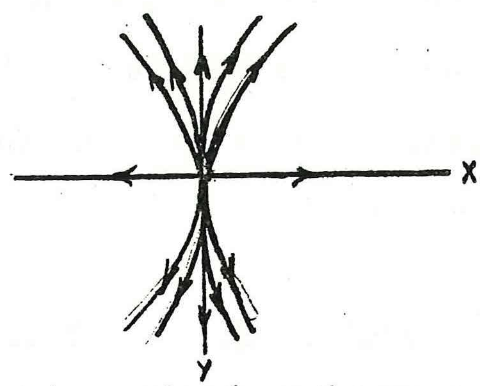


This last result says that the trajectories near a quadratic minimum look as pictured.

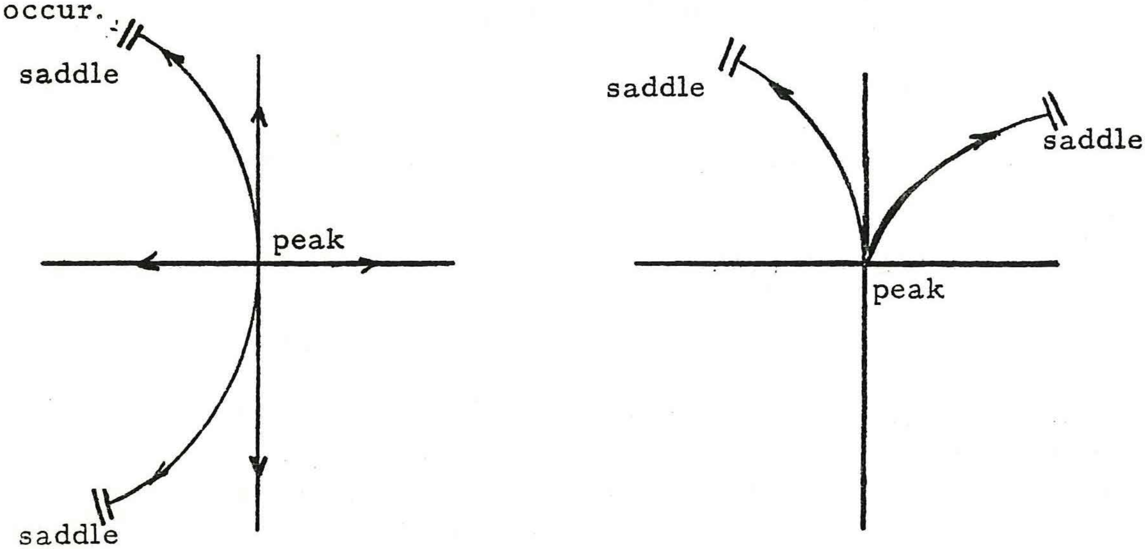
The same analysis may be applied to a quadratic maximum, i.e., a critical point where

$$\varphi^2 = Ax^2 + By^2$$

and $A, B < 0$. If $A < B$, then almost all trajectories leave the maximum tangent to the y-axis.



In particular, if there are two separatrices going from the same quadratic maximum to saddle points, the two situations pictured might occur.



In the first case, the two separatrices will together form a smooth curve. In the second case, a cusp is formed, so the separatrices do not form a submanifold.

From our previous analysis, we see that the question of whether there is a trajectory from a given maximum to a given minimum is decidable by a finite procedure -- just look at the components corresponding to each of the critical points. This is not true in a system more complicated than a gradient system. In certain cases, the decidability of this question depends on arithmetic properties of the coordinates; in some cases, the situation is, practically speaking, indeterminate.

42. Critical points in the degenerate situation.

So far a few results are known which apply to degenerate as well as non-degenerate singular points, but knowledge of the degenerate case remains sketchy compared to the non-degenerate case.

The theory of the Lusternik-Schnirelmann category states that there is a lower bound for the number $i = i(M)$ of <u>all</u> critical points of a manifold:[4)]

$$c(M^n) < i(M^n).$$

It has also been shown that on a manifold M^n of dimension n there is always a function with only $n+1$ critical points.[5)] This is

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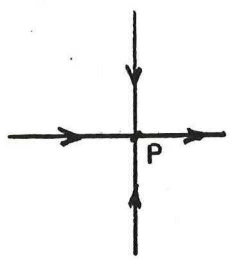
4) Liusternik - The topology of calculus of variations in the large, Translation of math. monographs, Amer. Math. Soc., 1966.

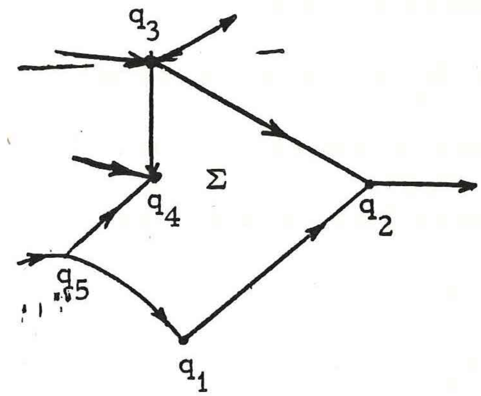
5) Èl'sgol'c, L. È. - Estimation of the number of critical points, Uspeki Matem. Nauk 5 n. 6(40), p. 52-87, 1950 (Russian) Math. Review p. 721, vol. 12, 1951.

similar to the result that M^n can be covered by n+1 charts! For example, in the case of a Riemannian manifold, we can do this in terms of the cut locus of a point $0 \in M$: For each $m \in M$, consider all geodesics joining 0 to m. There is one or more of shortest length. Let K_o be the set of points of M which have at least two such geodesics of shortest length. K_o is the cut locus of 0. $M-K_o$ is an open cell which is everywhere dense. If $x_o, \ldots, x_n$ are n+1 points of M in "general position", then $\bigcap K_{x_i} = \emptyset$, so the n+1 open cells $M-K_{x_i}$ cover M.

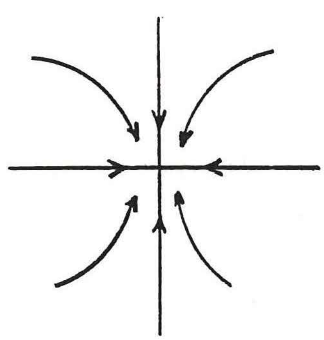
In studying a degenerate critical point p, we may use the following method. Perturb F to get a new function $F + \delta F$. By the Morse theorem, we may assume the new function has only non-degenerate critical points.. Thus we have a finite number of non-degenerate critical points of the new function, corresponding to the original degenerate critical point, to study.

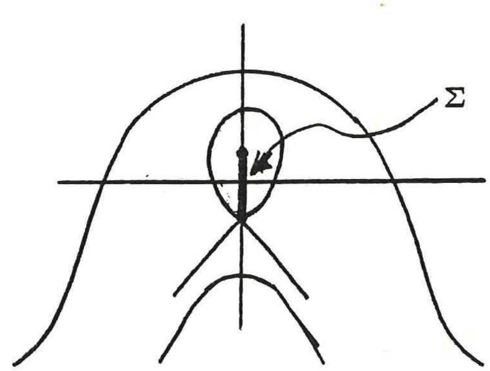
We can now state a conjecture as to the nature of degenerate critical points. Suppose the perturbation is also such that the gradient trajectories to and from the critical points always intersect transversally. The trajectories between two of the relevant critical points form a set Σ. The conjecture asserts that Σ is contractible and that the degenerate case arises by collapsing Σ, (i.e., that the sets Γ_p^- and Γ_p^+ of trajectories entering and leaving p are determined by the sets $\Gamma^-_{q_i}$ and $\Gamma^+_{q_i}$ for the q_i, perhaps by taking the disjoint union of the $\Gamma^{\pm}_{q_i}$ and identifying the points q_i). Then degenerate points could be characterized by the number of points and their indices in the non-degenerate approximation.





For example, let $f = x^3 + y^3$ in dimension 2. Perturb f to get $x^3 + y^3 - \lambda y$. For $\lambda > 0$, there will be two critical points. The new function gives the diagram on the right, which collapses to give the diagram for f on the left.





CHAPTER VI. FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

43. The Hamilton-Jacobi Equations.

In the first part, in §26 (page 118) we had occasion to consider the Hamilton-Jacobi Partial Differential Equation for a Hamiltonian function $H(q^1, \ldots, q^n, p_1, \ldots, p_n)$, and to show how suitable solutions of this equation determined the trajectories of the corresponding mechanical system. We now return to this topic, for the case when H also depends on the time (time-dependent Hamiltonian).

Let C be a configuration space, $M = T^*C$ the corresponding phase space. In the metric spaces $\mathbb{R} \times C$ and $\mathbb{R} \times M$ the projection onto $\mathbb{R}$ will be written as t, so that $t: \mathbb{R} \times M \to \mathbb{R}$ is the "time coordinate". Consider functions

$$H: \mathbb{R} \times M \to \mathbb{R} \quad , \quad S: \mathbb{R} \times C \to \mathbb{R}$$

(H is the hamiltonian, S the "entropy"). The function $S_t = S(t, -): C \to \mathbb{R}$ for each t determines a 1-form

$$dS_t: C \to M$$

d

(a cross section of the cotangent bundle). The corresponding equation

$$\frac{\partial S}{\partial t} + H(t, dS_t) = 0$$

is the (time-dependent) Hamilton-Jacobi equation.

Given the function S, each curve $b: I \to C$ lifts to a curve $c(t) = dS_t \circ b(t)$ in T^*C. The equations (for $q^1, \ldots, q^n$ coordinates in C)

$$\frac{dq^i b}{dt} = \frac{\partial H}{\partial p_i}(t, dS_t b) \quad , \quad i = 1, \ldots, n$$

will be called the <u>first Hamilton</u> equations, as they are the first half of the 2n Hamilton equations for C. The role of the Hamilton-Jacobi equation may be expressed by the following theorem (formulated by George Mackey):

<u>Theorem</u>. Let S be a solution of the Hamilton-Jacobi partial differential equation. Then if the path b in C satisfies the first Hamilton equation, the lifted path c satisfies Hamilton's equation. Conversely, let $S: \mathbb{R} \times C \to \mathbb{R}$ be a smooth function such that every b satisfying the first Hamilton equations yields a c satisfying (all of) Hamilton's equations. Then there is a smooth function $\psi: \mathbb{R} \to \mathbb{R}$ such that $S - \psi \circ t$ satisfies the Hamilton-Jacobi partial differential equation.

<u>Proof</u>. First we calculate

$$\begin{aligned}\frac{dp_i c}{dt} &= -\frac{d}{dt}\frac{\partial S}{\partial q^i}(t, q^1 b, \ldots, q^n b) \\ &= \frac{\partial^2 S}{\partial t\, \partial q^i} + \sum_j \frac{\partial^2 S}{\partial q^i \partial q^j}\frac{dq^j b}{dt} .\end{aligned}$$

By assumption, the first Hamilton equation holds. Hence

$$\frac{dp_i c}{dt} = \frac{\partial^2 S}{\partial t\, \partial q^i} + \sum_j \frac{\partial^2 S}{\partial q^i \partial q^j}\frac{\partial H}{\partial p_j} . \tag{1}$$

(Here each second partial of S has the evident arguments.)