

On the other hand, applying  $\frac{\partial}{\partial q^i}$  to Hamilton-Jacobi gives the following "derived H-J equation:

$$(2) \quad 0 = \frac{\partial^2 S}{\partial t \partial q^i} + \frac{\partial H}{\partial q^i}(t, dS_t) + \sum \frac{\partial H}{\partial p_j} \frac{\partial^2 S}{\partial q^j \partial q^i}$$

Now  $\frac{\frac{\partial S}{\partial q^j}}{\partial q^j \partial q^i} = \frac{\partial^2 S}{\partial q^i \partial q^j}$ . Hence subtracting the last two equations gives

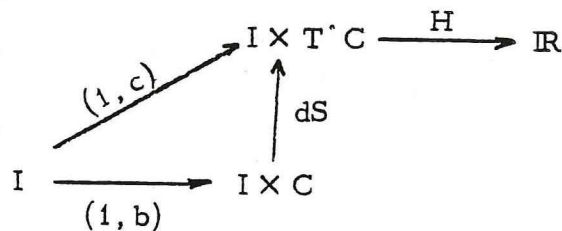
(with suitable arguments)

$$(3) \quad \frac{\partial p_{i,c}}{dt} = \frac{\partial H}{\partial q^i}$$

This is the second half of Hamilton's equations.

In the last equation everything is to be regarded as a function of  $t$ .

The "suitable arguments" required to make this the case are indicated without ambiguity by the diagram of the functions involved.



It should be possible to make a systematic use of such mapping diagrams to indicate which (composite) arguments we intended in equations (such as the Hamilton and Hamilton-Jacobi equations).

Now consider the converse part of the theorem. Take a solution  $b$  of the first Hamilton equations; then equations (1) above hold. By hypothesis (1) implies (3); subtracting, (2) holds along  $b$ . But by the existence theorem for ordinary differential equations, there is a solution  $b$  through

each point of  $I \times C$ ; hence (2) holds at any point  $(t, q^1, \dots, q^n)$ . But

(2) states that 
$$\frac{\partial}{\partial q^i} (HJ(S)) = 0, \quad i = 1, \dots, n,$$

where  $HJ(S)$  denotes the left-hand side of the Hamilton-Jacobi equations.

*ie HJ(S) depends on t, so*

Therefore, there is a smooth function  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  with

$HJ(S) = \theta \circ t: I \times C \rightarrow \mathbb{R}$ . Take a function  $\psi$  such that  $\frac{d\psi}{dt} = \theta$ . Then it is

not hard to see that

$$HJ(S - \psi) = 0.$$

This gives the conclusion of the theorem.

#### 44. Transformation to Equilibrium.

Now let  $Y$  be an  $n$ -dimensional manifold, and suppose

$S: \mathbb{R} \times C \times Y \rightarrow \mathbb{R}$  is a function such that everywhere

$$\det \left| \frac{\partial^2 S}{\partial q^i \partial y^j} \right| \neq 0$$

where  $q^i$  are coordinates for  $C$  and the  $y^j$  are coordinates for  $Y$ .

Let  $\mathbb{H}: \mathbb{R} \times C \times Y \rightarrow \mathbb{R} \times T^*C$  be given by

$$p_i^{\mathbb{H}} = \frac{\partial S}{\partial q^i}$$

$$t^{\mathbb{H}} = t$$

$$q^i{}^{\mathbb{H}} = q^i, \quad i = 1, \dots, n.$$

Thus the assumption on  $S$  is equivalent to saying that  $\mathbb{H}$  is regular

everywhere. Consider also the mapping  $\Psi: \mathbb{R} \times C \times Y \rightarrow \mathbb{R} \times T^*Y$

given by

$$\begin{aligned}
 t\Psi &= t \\
 y^i\Psi &= y^i \\
 x_i\Psi &= \frac{\partial S}{\partial y_i}, \quad i = 1, \dots, n,
 \end{aligned}$$

where the  $y^i$  are coordinates on  $Y$  and the  $x_i$  the corresponding (momentum) coordinates on  $T^*Y$ . This map is also regular everywhere.

Hence we have the diagram

$$\begin{array}{ccc}
 \mathbb{R} \times T^*C & \xleftarrow{\dots \chi \dots} & \mathbb{R} \times T^*Y \\
 \swarrow \textcircled{H} & & \nearrow \Psi \\
 \mathbb{R} \times C \times Y = N & & 
 \end{array}$$

and locally at least there is a map  $\chi$  from one time dependent phase space to another (compare §26).

Theorem. Take  $H: \mathbb{R} \times T^*C \rightarrow \mathbb{R}$ . For each point  $a \in Y$ , suppose that  $S: N \rightarrow \mathbb{R}$  satisfies the H-J partial differential equation for the function  $H$ . Let  $c: \mathbb{R} \rightarrow \mathbb{R} \times T^*Y$  be a curve of the form  $c(t) = (t, \text{const.})$ . Then the curve  $\chi c$  satisfies the Hamilton equations for  $H$  on  $\mathbb{R} \times T^*C$ .

Proof. It will be more convenient to look at everything in  $N$  rather than in  $\mathbb{R} \times T^*C$ . To do this we pull everything back locally by  $\textcircled{H}^{-1}$ . Thus we are interested in the curve  $\Psi^{-1}c$  and the function  $H\textcircled{H}$ .

Let  $x_i$  denote the coordinates as above. Then  $x_i = \frac{\partial S}{\partial y_i}$  on  $N$ .

Taking  $\frac{d}{dt}$  of this, we get

$$0 = \frac{\partial^2 S}{\partial y^i \partial t} + \sum \frac{\partial^2 S}{\partial y^i \partial q^j} \frac{dq^j}{dt} + \sum \frac{\partial^2 S}{\partial y^i \partial y^j} \frac{\partial y^j}{\partial t} .$$

