

C. How to get Riemannian or pseudo-Riemannian metrics.

(1) On one coordinate patch or on a Euclidean space, we have a system of vector fields $\partial_1, \dots, \partial_n$ valid everywhere. It suffices to define the functions $g_{ij}(x)$. We could, for example, set $g_{ij}(x) = \delta_{ij}$, making the vector fields $\{\partial_i\}$ orthonormal at each point.

(2) On a manifold, we can use (1) to construct metrics on coordinate patches. Then we can use "partitions of unity" to combine these metrics into a metric on the whole manifold. This is the method usually used to show that any manifold has a metric.

(3) The usual way in which Riemannian metrics arise in practice is as follows: Suppose N is a space (often a Euclidean space) which already has a Riemannian metric h . Suppose we have a manifold M and a C^∞ function $f: M \rightarrow N$ which is an immersion (that is, the Jacobian matrix of f is non-singular at every point of M). Then we define a metric $f^*(h) = g$ on M by letting $g_x(X, Y) = h_{f(x)}(f_*(X), f_*(Y))$ for $X, Y \in T_x M$.

In local coordinates, this goes as follows: Let x_1, \dots, x_m be local coordinates on M , and y_1, \dots, y_n be local coordinates on N . Then f is given by $y_i = f_i(x_1, \dots, x_m)$ ($i = 1, \dots, n$). Let h be given by $h_{ij}(y)$.

Then $g = f^*(h)$ is given by

$$\begin{aligned} g_{ij}(x) &= \sum_{k,l=1}^n h_{k,l}(f(x)) \frac{\partial f_k}{\partial x_i} \cdot \frac{\partial f_l}{\partial x_j} \\ &= \sum_{k,l=1}^n h_{k,l}(f(x)) \frac{\partial y_k}{\partial x_i} \cdot \frac{\partial y_l}{\partial x_j} \end{aligned}$$

$f^*(h)$ is called the pullback of h by f .

II (§49). Covariant Differentiation

A. Motivation. We want some sort of directional derivative on a manifold. Will L_X do? No! Why?

(1) A directional derivative in the direction X should depend only on the value of X at the point in question. But $L_{f \cdot X} Y = f \cdot L_X Y - (Y \cdot f)X$, showing that $L_X Y$ depends on how X is changing at the point in question (note the term $(Y \cdot f)X$).

(2) We will be interested in Newton's laws, and therefore in acceleration as we move along a curve; i. e., the derivative of the velocity vector in the direction of the velocity vector. But the Lie derivative $L_X X$ is always zero. Thus we cannot use L_X to discuss acceleration.

B. The abstract covariant derivative.

(1) Definition: An (affine) connection ∇ on M is a rule which assigns to two smooth vector fields X and Y on M another smooth vector field $\nabla_X Y$ on M , called the covariant derivative of Y in the direction X (with respect to ∇), obeying

$$(a) \nabla_{x_1 + x_2} Y = \nabla_{x_1} Y + \nabla_{x_2} Y \quad \text{and} \quad \nabla_{fX} Y = f \cdot \nabla_X Y,$$

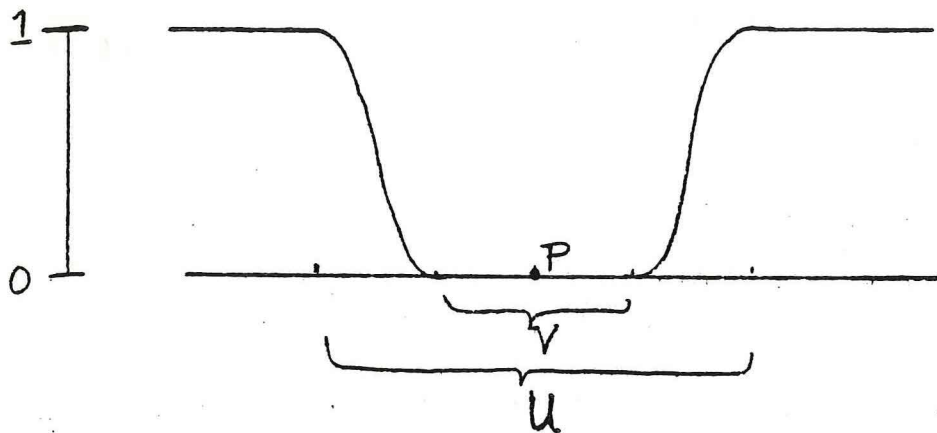
$$(b) \nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2 \quad \text{and} \quad \nabla_X (fY) = (X \cdot f)Y + f \cdot \nabla_X Y$$

for $f \in \mathcal{F}(M)$, X_i, Y_i vector fields on M .

(2) If $\nabla_X Y$ is to fulfill our expectations of what a directional derivative ought to be, then the following proposition should hold:

Proposition. For any point p on M , $(\nabla_X Y)_p$ depends only on X_p and on the behavior of Y in a neighborhood of p (actually on the "germ" of Y at p).

Proof. If $Y = Y'$ in a neighborhood U of p , then we take a "bump" function f which is one outside of U and zero on some neighborhood $V \subseteq U$ of p .



Then $Y - Y' = f \cdot (Y - Y')$, so that

$$\begin{aligned} (\nabla_X(Y - Y'))_p &= [\nabla_X(f \cdot (Y - Y'))]_p \\ &= (X \cdot f)_p (Y - Y')_p + f(p) \nabla_X(Y - Y')_p \\ &= 0 \cdot (Y - Y')_p + 0 \cdot \nabla_X(Y - Y')_p = 0, \end{aligned}$$

so $(\nabla_X Y)_p = (\nabla_X Y')_p$.

If $X_p = X'_p$, then we can write $X - X' = \sum f_i P_i$, where $f_i \in \mathcal{F}(M)$ and P_i are vector fields on M , with $f_i(p) = 0$. (Details left to the reader.) Then

